Solutions and answers to the homework problems from Sections 2.1-3.1

2.1.2. For (i) the equilibrium points are \( x = y = 0 \) and \( x = 10, y = 0 \). For the latter equilibrium point, prey alone exist and predators are absent. For (ii), the equilibrium points are \((0,0), (0,15)\), and \((3/5,30)\). For the latter equilibrium point, both species coexist. For \((0,15)\), the prey are extinct but predators survive.

Remark. This is a math problem and it doesn’t reflect the reality.

2.1.4. For (i), the prey obey a logistic model. The population tends to the equilibrium point at \( x = 10 \). For (ii), the prey obey the exponential growth model, so the population grows without limits.

2.1.6. For (i), the predators obey the exponential decay model, so the populations tends to 0. For (ii), the predators obey a logistic model. The population tends to the equilibrium point at \( y = 10 \).

2.2.10. (b) The solution is in the first quadrant and tends to the origin. The function \( x(t) \) initially increases, reaches a maximal value, and then approaches zero as \( t \to \infty \). It remains positive for all positive values of \( t \). The function \( y(t) \) decreases and tends to 0 as \( t \to 0 \).

2.2.12. The equilibrium solutions are those for which \( dR/dt = 0 \) and \( dF/dt = 0 \) simultaneously. To find the equilibrium points, we must solve the system of equations

\[
\begin{align*}
2R \left( 1 - \frac{R}{2} \right) - 1.2RF &= 0 \\
-F + 0.9RF &= 0
\end{align*}
\]

The second equations implies \( F = 0 \) or \( R = 10/9 \) and we consider each case independently (done in class). If \( F = 0 \) we find \( R = 0 \) or \( R = 2 \). If \( R = 10/9 \) then \( F = 20/27 \). The equilibrium points are \((R,F) = (0,0)\), \((R,F) = (2,0)\), and \((R,F) = (10/9,20/27)\).

2.2.22. (solved in class) (a) To find the equilibrium points we solve the following system of equations

\[
\begin{align*}
y(x^2 + y^2 - 1) &= 0 \\
-x(x^2 + y^2 - 1) &= 0
\end{align*}
\]

If \( x^2 + y^2 = 1 \), then both equations are satisfied. Hence, any point on the unit circle centered at the origin is an equilibrium point (infinitely many equilibria). If \( x^2 + y^2 \neq 1 \) then the first equation implies \( y = 0 \) and the second implies \( x = 0 \). Thus the origin \((0,0)\) is the only other equilibrium point.

(c) As \( t \) increases, typical solutions move on a circle around the origin, either counter-clockwise inside the unit circle, or clockwise outside the unit circle.
2.3.4. To check that \( \frac{dx}{dt} = 2x + 2y \) we compute both

\[
\frac{dx}{dt} = 4e^t + 4e^{4t}
\]

and

\[
2x + 2y = 8e^t + 2e^{4t} - 4e^t + 2e^{4t} = 4e^t + 4e^{4t}
\]

Similarly we verify that \( \frac{dy}{dt} = x + 3y \). Hence \( x(t), y(t) \) is a solution.

2.3.10. (a) Using the general solution

\[
\begin{pmatrix}
  k_2e^{2t} - \frac{k_1}{3}e^{-t}, k_1e^{-t}
\end{pmatrix},
\]

we easily see that \( k_1 = 3 \) and hence \( k_2 = 0 \). Therefore, we obtain

\[
Y(t) = (x(t), y(t)) = (-e^{-t}, 3e^{-t}).
\]

2.3.19. (solved in class) (a) For this system, we note that the second equation depends on \( y \) only. In fact this equation is both separable and linear, so we have a choice of two methods to find the general solution. The general solution is

\[
y(t) = -1 + k_1e^t.
\]

Substituting the above equation in the equation for \( \frac{dx}{dt} \) we find

\[
\frac{dx}{dt} = (-1 + k_1e^t)x.
\]

This equation is also both separable and linear and after solving it we find

\[
x(t) = k_2e^{-t+k_1e^t}.
\]

The general solution is

\[
(x(t), y(t)) = (k_2e^{-t+k_1e^t}, -1 + k_1e^t).
\]

(b) Setting \( \frac{dy}{dt} = 0 \) we obtain \( y = -1 \). From \( \frac{dx}{dt} = 0 \) we find \( xy = 0 \) and thus \( x = 0 \).

Therefore the equilibrium point is \( (x,y) = (0,-1) \).

(c) If \( (x(0), y(0)) = (1,0) \) we solve the following system

\[
k_2e^{-t+k_1e^t} = 1 \\
-1 + k_1e^t = 0.
\]

Hence \( k_1 = 1 \) and \( k_2 = 1/e = e^{-1} \). Thus the solutions is

\[
(x(t), y(t)) = (e^{-1}e^{-t+e^t}, -1 + e^t) = (e^{-1-t+e^t}, -1 + e^t).
\]

3.1.8.

\[
\begin{align*}
\frac{dx}{dt} & = -3x + 2\pi y \\
\frac{dy}{dt} & = 4x - y
\end{align*}
\]
3.1.24. (a) Substituting $Y_1(t)$ into the left-hand side of the system yields

$$\frac{dY_1}{dt} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}.$$  

The right-hand side becomes

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^t \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}. $$

Since the two sides agree, $Y_1(t)$ is a solution.

Similarly, if we substitute $Y_2(t)$ into the left-hand side, we obtain

$$\frac{dY_2}{dt} = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix}.$$  

The right-hand side becomes

$$\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ 2e^{2t} \end{bmatrix}. $$

Since the two side agree, $Y_2(t)$ is also a solution.

(b) Since by the Linearity Principle any linear combination of solutions is also a solution, we seek the solution satisfying the initial condition $Y(0) = (-2, -1)$ in the form

$$Y(t) = c_1 Y_1(t) + c_2 Y_2(t).$$

To compute the constants $c_1, c_2$, we take $t = 0$ and obtain

$$Y(0) = c_1 Y_1(0) + c_2 Y_2(0)$$

which is the same as

$$\begin{bmatrix} -2 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}. $$

This vector equation is equivalent to the system of linear equations

$$c_2 = -2$$

$$c_1 + c_2 = -1.$$  

From the first equation, $c_2 = -2$. Substituting into the second equation, we obtain $c_1 = 1$. Therefore, the solution satisfying $Y(0) = (-2, -1)$ is

$$Y(t) = Y_1(t) - 2Y_2(t)$$

$$= \begin{bmatrix} 0 \\ e^t \end{bmatrix} - 2 \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{2t} \\ e^t - 2e^{2t} \end{bmatrix}.$$  

We can check that this is indeed a solution by substituting into the system. Then both sides become

$$\begin{bmatrix} -4e^{2t} \\ e^t - 4e^{2t} \end{bmatrix}.$$
which means that $Y(t)$ is a solution.

**Remark.** This problem can be solved also with the technique of eigenvalues and eigenvectors discussed in Section 3.2 (for example, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$).

**3.1.28.** (a) We verify that $Y_1(t)$ and $Y_1(t)$ are solutions. That is done in class and the approach is like the one in the solution of the last problem.

(b) Note that $Y_1(0) = (1,0)$ and $Y_2(0) = (0,1)$, and these vectors are not on the same line through the origin. Hence, $Y_1(t)$ and $Y_2(t)$ are linearly independent.

(c) We must find constants $k_1$ and $k_2$ for which

$$k_1 Y_1(0) + k_2 Y_2(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$ 

We have then $k_1 = 2$ and $k_2 = 3$, and the solution is

$$Y(t) = e^{-2t} \begin{bmatrix} 2 \cos 3t - 3 \sin 3t \\ 2 \sin 3t + 3 \cos 3t \end{bmatrix}.$$ 

**Remark.** This problem can not be solved with the technique of eigenvalues and eigenvectors discussed in Section 3.2 as the eigenvalues are not real.