Answers of some homework problems 2.1-2.8

2.1.4. (b) The slope is \( \frac{1}{2} \).
(c) The equation is \( y - \ln 2 = \frac{1}{2} (x - 2) \).

2.1.6. The average velocity between \( t \) and \( t + h \) seconds is
\[
\frac{58(t + h) - 0.83(t + h)^2 - (58t - 0.83t^2)}{h} = \ldots = 58 - 1.68t - 0.83h, \ h \neq 0.
\]
(a) Here \( t = 1 \), so for (i) we have \([1, 2] \): \( h = 1, 55.51 \text{ m/s, etc.} \)
(b) The instantaneous velocity after 1 second is 56.34 m/s.

2.2.16. The limit \( \lim_{x \to -1} \frac{x^2 - 2x}{x^2 - x - 2} \) does not exist.

2.2.24. We have \( \lim_{x \to 5^-} 6x - 5 = -\infty \) since \( (x - 5) \to \infty \) as \( x \to 5^- \) and \( \frac{6}{x - 5} < 0 \).

2.2.30. We have \( \lim_{x \to -5^+} \ln(x - 5) = -\infty \) since \( x - 5 \to 0^+ \) as \( x \to 5^+ \).

2.3.6. After applying the corresponding Limit Laws we find that the limit is 256.

2.3.14. We find
\[
\lim_{x \to 4} \frac{x^2 - 4x}{x^2 - 3x - 4} = \lim_{x \to 4} \frac{x}{x + 1} = \frac{4}{5}.
\]

2.3.22. We have
\[
\lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h} = \lim_{h \to 0} \frac{\sqrt{1 + h} - 1}{h} \cdot \frac{\sqrt{1 + h} + 1}{\sqrt{1 + h} + 1} = \ldots = \lim_{h \to 0} \frac{1}{\sqrt{1 + h} + 1} = \frac{1}{2}.
\]

2.3.26. We have
\[
\lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 + 1} \right) = \lim_{t \to 0} \frac{t}{t(t^2 + 1)} = \lim_{t \to 0} \frac{1}{t + 1} = 1.
\]

2.3.30. Solved in class.

2.3.38. Since \(-1 \leq \sin(\pi/x) \leq 1\) we have
\[
\sqrt{x}/e \leq \sqrt{x}e^{\sin(\pi/x)} \leq \sqrt{x}e.
\]
Now, the Squeeze Theorem implies \( \lim_{x \to 0} \sqrt{x}e^{\sin(\pi/x)} = 0 \).

2.3.42. The limit does not exist (solved in class).

2.4.20. Given \( \epsilon > 0 \) we must find \( \delta > 0 \), such that, if \( 0 < |x - 6| < \delta \), then \( \left| \frac{x}{4} + 3 - \frac{x}{2} \right| < \epsilon \). Equivalently we have that \( 0 < |x - 6| < \delta \) implies \( |x - 6| < 4\epsilon \). So, we choose \( \delta := 4\epsilon \). Then... (as in class).
2.4.26. Given \( \varepsilon > 0 \) we must find \( \delta > 0 \), such that, if \( 0 < |x - 60| < \delta \), then \( |x^3 - 0| < \varepsilon \). Then (solved in class) choose \( \delta := \sqrt[3]{\varepsilon} \).

2.4.28. Given \( \varepsilon > 0 \) we must find \( \delta > 0 \) such that if \( 9 - \delta < x < 9 \) then \( |\sqrt{9} - x - 0| < \varepsilon \). Also solved in class: we choose \( \delta := \varepsilon^4 \).

2.4.30. Solved in class.

2.5.12. We have
\[
\lim_{x \to 4} g(x) = \lim_{x \to 4} \frac{x + 1}{2x^2 - 1} = ... = \frac{5}{31} = g(1).
\]
Therefore, \( g \) is continuous at \( a = 4 \).

2.5.18. We find that
\[
\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - x}{x^2 - 1} = ... = \frac{1}{2} \neq f(1).
\]
Therefore, \( f \) is discontinuous at \( a = 1 \).

2.5.20. We have \( \lim_{x \to 1^-} f(x) = 2 \) and \( \lim_{x \to 1^+} f(x) = 3 \). Therefore, \( f \) is discontinuous at \( a = 1 \), because \( \lim_{x \to 1} f(x) \) does not exist.

2.5.22. The root function \( \sqrt{x} \) and the polynomial function \( 1 + x^3 \) are continuous on \( \mathbb{R} \). Therefore (part 4 of Theorem 4 in the book), the product \( G(x) = \sqrt{x}(1 + x^3) \) is continuous on its domain \( \mathbb{R} \).

2.5.26. By Theorem 5 in the book, the polynomial \( x^2 - 1 \) is continuous on \( \mathbb{R} \). By Theorem 7 in the book, the inverse trigonometric function \( \sin^{-1} \) is continuous on \([ -1, 1 ]\). Therefore (Theorem 9 in the book), the composition \( \sin^{-1}(x^2 - 1) \) is continuous on its domain \([ -\sqrt{2}, \sqrt{2} ]\).

2.5.38. The function \( f \) is continuous at every point except at 1 and 3. Although it is discontinuous at 3, \( f \) is continuous from the right at 3. Furthermore, \( f \) is continuous from the right at 1.

2.6.16. Dividing the numerator and the denominator by \( x^2 \) we find
\[
\lim_{y \to \infty} \frac{2 - 3y^2}{5y^2 + 4y} = \lim_{y \to \infty} \frac{2/y^2 - 3}{5 + 4/y} = ... = \frac{-3}{5}.
\]

2.6.20. Dividing the numerator and the denominator by \( x \) we find
\[
\lim_{x \to \infty} \frac{x + 2}{\sqrt{9x^2 + 1}} = \lim_{x \to \infty} \frac{1 + 2/x}{\sqrt{9 + 1/x^2}} = \frac{1}{3}.
\]

2.6.24. Solved in class. The answer is \(-1\). Keep in mind (me too!) that if \( x \to -\infty \), then \( \sqrt{x^2} = -x \).

2.6.26. The limit \( \lim_{x \to \infty} \cos x \) does not exist.
2.6.28. We have that $\sqrt[3]{x}$ is arbitrarily small (large negative) as $x$ is sufficiently small (large negative). Thus $\lim_{x \to -\infty} \sqrt[3]{x} = -\infty$.

2.6.30. Dividing the numerator and the denominator by $x^3$ we find

$$\lim_{x \to -\infty} \frac{x^3 - 2x + 3}{5 - 2x^2} = \lim_{x \to -\infty} \frac{x - 2x + 3/x^3}{5/x^2 - 2} = -\infty,$$

because the numerator of the last limit approaches $\infty$ and the denominator approaches $-2$ as $x \to -\infty$.

2.6.38. The vertical asymptotes are $x = 1$ and $x = -1$. The horizontal asymptote is $y = 1$.

2.6.42. The horizontal asymptotes are $y = -\frac{1}{2}$ and $y = \frac{1}{2}$. There are no vertical asymptotes.

2.6.64. Given $M > 0$ we must find $N > 0$ such that if $x > N$ then $x^3 > M$. As we solved in class, we choose $N := \sqrt[3]{M}$.

2.7.8. We find that

$$m = \lim_{x \to 4} \frac{\sqrt{2x + 1} - \sqrt{2(4) + 1}}{x - 4} = \lim_{x \to 4} \frac{\sqrt{2x + 1} - 3}{x - 4} \cdot \frac{\sqrt{2x + 1} + 3}{\sqrt{2x + 1} + 3} = \frac{1}{3}.$$

Therefore, the tangent line is $y - 3 = \frac{1}{3}(x - 4)$.

2.7.10. We find that

$$m = \lim_{x \to 0} \frac{2x}{x + 1} - \frac{0}{x - 0} = \lim_{x \to 0} \frac{2}{(x + 1)^2} = 2.$$

Therefore the tangent line is $y - 0 = 2(x - 0)$, or $y = 2x$.

2.7.18. (b) We have

$$v(a) = \lim_{h \to 0} \frac{H(a + h) - H(a)}{h} = \lim_{h \to 0} \frac{58a + 58h - 0.83a^2 - 1.66ah - 0.83h^2 - (58a - 0.83a^2)}{h} = 58 - 1.66a$$

(c) $t = \frac{58}{0.83}$

(d) $v(\frac{58}{0.83}) = -58$ m/s.

2.7.20. The average velocity between times $t$ and $t + h$ is

$$\frac{s(t + h) - s(t)}{(t + h) - t} = \ldots = (2t + h - 8) \text{ m/s}$$

(i) $[3,4]: t = 3, h = 1$, so the average velocity is $2(3) + 1 - 8 = -1$ m/s...

(b) $v(t) = 2t - 8, v(4) = 0$.

2.8.13. We have

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{3(a + h) + 4(a + h)^2}{h} = -2 + 8a.$$
2.8.14. We have
\[ f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{((a + h)^4 - 5(a + h)) - (a^4 - 5a)}{h} = ... = 4a^3 - 5. \]
Here we use the fact that \( x^4 - y^4 = (x^2 - y^2)(x^2 + y^2) = (x - y)(x + y)(x^2 + y^2) \).

2.8.16. We have
\[ f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{(a + h)^{2+1} - a^{2+1}}{h - a - 2} = ... = \frac{a^2 - 4a - 1}{(a - 2)^2}. \]

2.8.18. We have
\[ f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{\sqrt{3(a + h)} - 1 - \sqrt{3a - 1}}{h} = ... = \frac{3}{2\sqrt{3a + 1}}. \]

2.8.20. The answer is \( f'(a) \), where: \( f(x) = \sqrt[3]{x}, a = 16 \).

2.8.22. The answer is \( f'(a) \), where: \( f(x) = \tan x, a = \frac{\pi}{4} \).

2.8.24. The answer is \( f'(a) \), where: \( f(t) = t^4 + t, a = 1 \).