Notes on limits and continuity
Math 112

In this handout, we take another look at limits that may make the differences between the one- and two-variable situations a little clearer. To be somewhat formal for a moment, the definition of limit in terms of sequences is:

**Definition.** We say that a sequence $x = (x_1, x_2, \ldots)$ in $\mathbb{R}^n$ approaches a point $a$ in $\mathbb{R}^n$ if, given any neighborhood $D_r(a)$ of $a$, $x$ is eventually contained in $D_r(a)$ (i.e., all but many points of $x$ are contained in $D_r(a)$).

**Definition.** The function $f : A \subseteq \mathbb{R}^2 \to \mathbb{R}$ has a limit $L$ at the point $(a, b)$, written

$$\lim_{(x, y) \to (a, b)} f(x, y) = L,$$

if, for any sequence $(x, y)$ in $A$ that approaches, but never reaches, $(a, b)$ along any route/path, the value of $f(x, y)$ must approach $L$. In other words, to say that $\lim_{(x, y) \to (a, b)} f(x, y) = L$ means that if $(x, y) \to (a, b)$ with $(x, y)$ never equal to $(a, b)$, then we must have $f(x, y) \to L$.

The technical aspects of the term “approach” aside, hopefully the main idea is clear: For $f(x, y)$ to have a limit $L$ at $(a, b)$, the values of $f(x, y)$ must approach $L$ as $(x, y)$ approaches (but does not reach) $(a, b)$, no matter how $(x, y)$ approaches $(a, b)$. Note that it follows that if we can find at least one way to approach $(a, b)$ along which the values of $f$ do not approach $L$, then $f$ cannot have a limit of $L$ at $(a, b)$.

Here’s an important example to keep in mind. Let $f$ be defined by:

$$f(x, y) = \begin{cases} 1 & \text{if } 0 < y < x^2; \\ 0 & \text{otherwise.} \end{cases}$$

Note that the inequalities in the definition of $f$ are all “less than” and not “less than or equal to.”

The function $f$ is perhaps best understood from its contour diagram:
Here, the shaded region is where $f(x, y) = 1$, and the unshaded region (including the
curve $y = x^2$ and the entire $x$-axis) is where $f(x, y) = 0$.

Now, since $f(0, 0) = 0$, if $f$ were continuous at $(0, 0)$, then $f(x, y)$ would have to
approach 0 as $(x, y)$ approaches the origin by any route. However, if $(x, y)$ approaches
the origin by a route that stays in the shaded region (e.g., along the curve $y = x^2/2$),
then $f(x, y)$ remains constant at 1 and therefore does not approach 0. It follows that
$f$ is not continuous at $(0, 0)$. In fact, since we may also approach the origin by a
route along which $f(x, y) = 0$ (e.g., along the $y$-axis), we can actually conclude that
\[
\lim_{(x, y) \to (0,0)} f(x, y)
\]
does not exist, as such a limit would have to be both 0 and 1.

On the other hand, note that if we approach the origin along \textit{any} straight-line
path, then the value of $f(x, y)$ will eventually become 0 at some distance away from
the origin (why?), so if we only consider straight-line approaches, it looks like $f(x, y)$
is continuous. Again, the fundamental point is that to understand a function of two
variables at a point $(a, b)$, it is not enough to look at $f(x, y)$ as $(x, y)$ approaches $(a, b)$
along (for instance) straight lines; you have to look at what happens to $f(x, y)$ as
$(x, y)$ approaches $(a, b)$ in an \textit{arbitrary} (straight, curved, spiral, etc.) way. Later on in
Chapter 2, we will see that this is precisely the limitation of looking at multivariable
functions one variable at a time: namely, that change can occur along paths that
cannot be understood by considering only one variable at a time. To repeat:

\textbf{It is not enough to consider one variable at a time.}