
Problems to be turned in:

1. Let \( F : [a, b] \to \mathbb{R} \) be differentiable, and suppose that \( \frac{dF}{dx} \) is continuous on \( [a, b] \).
   
   (a) Let \( G(x) = \int_a^x F'(t) \, dt \), and let \( H(x) = F(x) - G(x) \). Find the value of \( H'(x) \).
   
   What conclusion can you draw, and why?

   (b) Prove that
   
   \[
   F(b) - F(a) = \int_a^b \frac{dF}{dx} \, dx.
   \]
   
   Suggestion: What is \( H(a) \)?

2. Let

   \[
   f(x) = 100x(x-1)(x-2)e^{-x^6}.
   \]

   Numerical integration shows that (rounded off)
   
   \[
   \int_0^1 f(x) \, dx \approx 23.56, \quad \int_1^2 f(x) \, dx \approx -0.32, \quad 0 < \int_2^\infty f(x) \, dx < 10^{-7},
   \]

   where by definition, \( \int_2^\infty f(x) \, dx = \lim_{b \to \infty} \int_2^b f(x) \, dx \).

   Let \( F : [0, \infty) \to \mathbb{R} \) be a differentiable function such that \( F'(x) = f(x) \) and \( F(0) = 13 \).

   (a) Does \( F(x) \) attain an absolute maximum value? If so, for which value(s) of \( x \)? Prove your answer.

   (b) Does \( F(x) \) attain an absolute minimum value? If so, for which value(s) of \( x \)? Prove your answer.

   Suggestion: Do not try to compute a formula for the indefinite integral of \( f(x) \).

3. Let \( f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{2^n \sqrt{n}} \).

   (a) Find the radius of convergence of \( f(x) \).

   (b) Find the exact interval of convergence of \( f(x) \). (I.e., what happens on the boundary?)

   (Cont. on other side.)
4. Let
\[ S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}. \]

(a) Prove that \( S(x) \) and \( C(x) \) converge for all \( x \in \mathbb{R} \).

(b) Explain why that implies that the radii of convergence of \( S(x) \) and \( C(x) \) are both equal to \( \infty \). Warning: Keep Example 6 of Sect. 23 (p. 190) in mind.

5. Ex. 23.8.


7. REVISED THU APR 30, 3:30pm: Change \([0, 1]\) to \((0, 1)\). That is:

Let \( f_n(x) = \sum_{k=0}^{n} x^k \).

(a) Does the sequence \( (f_n) \) converge pointwise on the set \((0, 1)\)? If so, give the limit function. (Suggestion: See p. 96.)

(b) Does \( (f_n) \) converge uniformly on \((0, 1)\)? Prove your assertion. (Suggestion: Use Remark 24.4/problem 8, below.)

8. Let \( S \subseteq \mathbb{R} \) be nonempty, and let \( f, f_n : S \rightarrow \mathbb{R} \) be functions. For \( n \in \mathbb{N} \), let \( D_n = \sup \{|f(x) - f_n(x)| \mid x \in S\} \). Prove that the following are equivalent (if and only if):

- \( f_n \) converges uniformly to \( f \) on \( S \).
- \( \lim_{n \to \infty} D_n = 0. \)