## Midterm 1 Solutions

Section 04

<table>
<thead>
<tr>
<th></th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>25</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
</tr>
</tbody>
</table>
1. **(25 points)** Consider a differential equation

\[ yy' - \frac{2x}{\cos y} = 0. \]  

(1)

(a) If \( x_0 \neq 0 \), show that (1) cannot possibly have a solution defined in a neighborhood of \( x_0 \) that satisfies \( y(x_0) = 0 \).

(b) Find an implicit solution satisfying \( y(0) = 0 \).

**Solution:**

(a) If \( y \) were a solution to (1) satisfying \( y(x_0) = 0 \), then since \( \cos 0 = 1 \),

\[ 0 \cdot y'(x_0) - 2x_0 = 0, \]

which is impossible if \( x_0 \neq 0 \).

(b) Separating the variables we obtain

\[ \int y \cos y \, dy = \int 2x \, dx. \]

The right hand side equals \( x^2 + C \). The left hand side can be solved by integration by parts: \( u = y, dv = \cos y, du = dy, v = \sin y \):

\[ \int y \cos y \, dy = y \sin y - \int \sin y \, dy = y \sin y + \cos y. \]

Setting \( y(0) = 0 \), we obtain

\[ C = 1. \]

Therefore, an implicit solution satisfying \( y(0) = 0 \) is

\[ y \sin y + \cos y = x^2 + 1. \]
2. (25 points) Consider a differential equation
\[(3x^2 + y) \, dx + (x^2y - x) \, dy = 0.\] (2)

(a) Show that equation (2) is not exact.
(b) Find an integrating factor \(\mu\) that depends only on \(x\) or only on \(y\), and solve the equation.

Solution: (a) Since
\[
\frac{\partial}{\partial y}(3x^2 + y) = 1 \neq 2xy - 1 = \frac{\partial}{\partial x}(x^2y - x),
\]
the equation is not exact.

(b) Let’s try to find an integrating factor \(\mu = \mu(x)\). Let
\[
\tilde{M}(x, y) = \mu(x)(3x^2 + y), \quad \tilde{N}(x, y) = \mu(x)(x^2y - x).
\]
We have
\[
\frac{\partial \tilde{M}}{\partial y} = \mu(x) \quad \text{and} \quad \frac{\partial \tilde{N}}{\partial x} = \mu'(x)(x^2y - x) + \mu(x)(2xy - 1).
\]
Requiring \(\partial \tilde{M}/\partial y = \partial \tilde{N}/\partial x\) gives us a differential equation for \(\mu\):
\[
\mu(x) = \mu'(x)(x^2y - x) + \mu(x)(2xy - 1).
\]
Simplifying, we obtain
\[
\mu'(x) = -\frac{2\mu(x)}{x},
\]
so
\[
\mu(x) = x^{-2}.
\]
Multiplying the original equation by \(\mu(x)\), we get an exact equation:
\[
\left(3 + \frac{y}{x^2}\right) \, dx + \left(y - \frac{1}{x}\right) \, dy = 0.
\]
Integration of \(\tilde{M}\) with respect to \(x\) yields
\[
F(x, y) = 3x - \frac{y}{x} + g(y),
\]
and a differentiation with respect to \(y\) gives us
\[
g'(y) = y.
\]
So \(g(y) = y^2/2\) and the desired implicit solution to (2) is
\[
F(x, y) = 3x - \frac{y}{x} + \frac{y^2}{2} \equiv C.
\]
Note that the \(y\)-axis \(x = 0\) is also a solution curve for (2).
3. (25 points) Using an appropriate substitution, find the solution to

\[ y' + y = e^x y^{-2} \]

satisfying

\[ y(0) = \left( \frac{3}{4} \right)^{1/3}. \]

Solution: This is a Bernoulli equation with \( n = -2 \). The appropriate substitution is

\[ v = y^{1-n} = y^3. \]

Then

\[ v' = 3y^2 y'. \]

If we multiply the original equation by \( 3y^2 \), we obtain

\[ 3y^2 y' + 3y^3 = 3e^x, \]

which is equivalent to a linear equation

\[ v' + 3v = 3e^x. \]

The integrating factor for this equation is

\[ \mu(x) = e^{\int P(x) \, dx} = e^{3x}. \]

Therefore,

\[
\begin{align*}
v &= e^{-3x} \left( \int e^{3x} \cdot 3e^x \, dx + C \right) \\
  &= \frac{3}{4}e^x + Ce^{-3x}.
\end{align*}
\]

Taking the cube root, we get

\[ y = v^{1/3} = \left( \frac{3}{4}e^x + Ce^{-3x} \right)^{1/3}. \]

Since \( y(0) = (3/4)^{1/3} \), we obtain \( C = 0 \), so the solution to the initial value problem is

\[ y = \left( \frac{3}{4}e^x \right)^{1/3}. \]
4. (25 points) (a) Show that all curves in the one-parameter family
\[ x^2 + Cy^2 = 1 \]  \hspace{1cm} (3)
are implicit solutions to the differential equation
\[ y' = \frac{xy}{x^2 - 1}. \]

(b) Write a differential equation satisfied by the curves orthogonal to the family (3). Find an (algebraic) equation for those curves.

Solution: (a) Differentiating implicitly we get
\[ 2x + 2Cyy' = 0. \]
If we solve for \( y' \), we obtain
\[ y' = -\frac{2x}{2Cy} = -\frac{xy}{Cy^2} = -\frac{xy}{1-x^2} = \frac{xy}{x^2 - 1}. \]

(b) The slope of the orthogonal curves is the negative reciprocal of the slope of the original curves, so the differential equation for the orthogonal curves is
\[ y' = -\frac{x^2 - 1}{xy}. \]

Separating the variables we obtain
\[ \int y \, dy = \int \left( \frac{1}{x} - x \right) \, dx, \]
\[ \frac{y^2}{2} = \log|x| - \frac{x^2}{2} + C. \]