Midterm 1 Solutions

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1. **(25 points)** (a) Write the equation of a line containing the points \((1, \frac{8}{5})\) and \((0, 2)\).

   (b) What is its slope?

   (c) Does the line contain the point \((2, 1)\)?

**Solution:** (a) We’ll look for an equation in the form \(y = mx + b\). Since the line contains \((0, 2)\), we have

\[
2 = m \cdot 0 + b = b,
\]

and since it contains \((1, \frac{8}{5})\), we obtain

\[
\frac{8}{5} = m + 2,
\]

implying \(m = -2/5\). Therefore, the equation is

\[
y = -\frac{2}{5}x + 2, \quad \text{or} \quad 2x + 5y = 10.
\]

(b) The slope is \(m = -2/5\).

(c) No, it does not contain the point \((2, 1)\), because

\[
2 \cdot 2 + 5 \cdot 1 \neq 10.
\]
2. **(25 points)** A high school football team with 40 players includes 16 players who played offense last year, 17 who played defense, and 12 who were not on last year’s team. How many players from last year played both offense and defense?

**Solution 1:** Let $O, D$ denote the sets of players who played offense and defense last year, respectively. Then $n(O) = 16$, $n(D) = 17$. The players who were on last year’s team form the set $O \cup D$, so those who weren’t, form its complement, $(O \cup D)'$, which has 12 elements. Therefore,

$$n(O \cup D) = 40 - n((O \cup D)') = 40 - 12 = 28.$$  

The Additive Principle of counting says

$$n(O \cup D) = n(O) + n(D) - n(O \cap D),$$

so the number of players who played both offense and defense last year is

$$n(O \cap D) = n(O) + n(D) - n(O \cup D) = 16 + 17 - 28 = 5.$$ 

**Solution 2:** On this year’s team there are $28 = 40 - 12$ people who were on last year’s team. Since 16 of them played offense, 17 played defense, and $16 + 17 = 33$, $33 - 28 = 5$ must have played both offense **and** defense.
3. (25 points) A basketball team has 5 distinct positions. Out of 8 players, how many starting teams are possible if

(a) The distinct positions are taken into consideration?
(b) The distinct positions are not taken into consideration, but either Mike or Ken (but not both) must start?

Solution: (a) Since order is clearly important here, the answer is the number of permutations of 8 elements (players) taken 5 at a time:

\[ P_{8,5} = \frac{8!}{(8-5)!} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4. \]

Alternatively, there are 8 possible choices for the first position, 7 for the second, 6 for the third, etc., and the answer is the same.

(b) If Mike starts (but not Ken), then there are 6 players (8 minus Mike minus Ken) to fill 4 positions. Since order is not important, the number of ways to fill them is the number of combinations of 6 elements taken 4 at a time:

\[ C_{6,4} = \frac{6!}{4!(6-4)!} = \frac{6 \cdot 5}{2 \cdot 1} = 15. \]

If Ken is to start (but not Mike), the number of ways to fill the remaining 4 positions by the remaining 6 players is again \( C_{6,4} \). Therefore, if either Mike or Ken is to start (but not both), there are

\[ 2C_{6,4} = 30 \]

starting teams.
4. **(25 points)** An experiment consists of rolling two fair dice and adding the dots on the two sides facing up. Assuming that each simple event is as likely as any other, find the probability that the sum is 7 or 11 (a “natural”)?

**Solution:** The sample space $S$ is the set of all ordered pairs $(i, j)$, where $i$ and $j$ are numbers $1, 2, 3, 4, 5, 6$. Therefore, $n(S) = 36$.

Let $E_7$ be the event that the sum is 7 and $E_{11}$ the event that the sum is 11. Then

$$E_7 = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\},$$

$$E_{11} = \{(5, 6), (6, 5)\}.$$

The event that the sum is 7 or 11 is

$$E = E_7 \cup E_{11}.$$

Since $E_7$ and $E_{11}$ are disjoint (the sum can’t be both 7 and 11), $n(E) = n(E_7) + n(E_{11}) = 6 + 2 = 8$. Therefore, the probability that the sum is 7 or 11 is

$$P(E) = \frac{n(E)}{n(S)} = \frac{8}{36} = \frac{2}{9}.$$