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Explain your work
1. **(20 points)** Consider the differential equation on the real line

\[ x' = x^2 \sin x. \]

Find the equilibria, determine their type, and draw the phase line.

**Solution:** The equilibria are \( x_n = n\pi \), for all integers \( n \).

The sign of \( x' = x^2 \sin x \) is the same as the sign of \( \sin x \), which is positive for \( x \in (2k, 2k + 1) \) and negative for \( x \in (2k + 1, 2k + 2) \) (\( k \in \mathbb{Z} \)). Therefore, \( x_{2k} \)'s are sources and \( x_{2k+1} \)'s are sinks.

The phase line therefore looks like this:

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Figure 1: The phase line.
2. (20 points) Consider the linear system $X' = AX$, where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

(a) Find the eigenvalues and eigenvectors.

(b) Find a matrix $T$ that puts $A$ into canonical form, $B$.

(c) Sketch the phase portraits of $X' = AX$ and $Y' = BY$.

Solution: (a) The eigenvalues are $\lambda_1 = -1, \lambda_2 = 1$. The corresponding eigenvectors $V_i = (x_i, y_i) \ (i = 1, 2)$ are solutions to $(A - \lambda_i I)V_i = 0$, i.e.,

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0,$$ 

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0.$$ 

We obtain $x_1 = -y_1$ and $x_2 = y_2$, so we can take $V_1 = (1, -1)$ and $V_2 = (1, 1)$.

(b) We have

$$T = [V_1|V_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \text{and} \quad B = T^{-1}AT = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

(c) The phase portraits are:

![Phase portraits](image)

Figure 2: Phase portraits of $X' = AX$ and $Y' = BY$. 
3. **(20 points)** Given are linear systems $X' = A_k X$ ($k = 1, \ldots, 4$), where

$$A_1 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -3 \\ -1 & 3 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -2 & -1 \\ 2 & -3 \end{bmatrix}. $$

Determine which systems are topologically conjugate to each other and explain why.

**Solution:** The eigenvalues of $A_1$ and $A_3$ are $\{0, 2\}$ and $\{0, 4\}$. By Homework 4, $X' = A_1 X$ and $X' = A_3 X$ are topologically conjugate.

Both $A_2$ and $A_4$ have negative trace (equal to $-5$) and positive determinant (5 and 8, respectively). Therefore, they both have two eigenvalues with negative real part (in fact, $A_2$ has two real negative eigenvalues, and $A_4$ complex conjugate ones), which implies that $X' = A_2 X$ and $X' = A_4 X$ are topologically conjugate.

Observe that if $i \in \{1, 3\}$ and $j \in \{2, 4\}$, then $X' = A_i X$ in **not** topologically conjugate to $X' = A_j X$. 

4. (20 points) Consider the following system of differential equations:

\[
\begin{align*}
x' &= x^2 - 1 \\
y' &= -y.
\end{align*}
\]

(a) Find the equilibria and determine their type using linearization.

(b) Show that the lines \(x = -1\), \(x = 1\), and \(y = 0\) are invariant, and approximately sketch the phase portrait.

**Solution:**

(a) The equilibria are \(X_- = (-1, 0)\) and \(X_+ = (1, 0)\). Denote the vector field of the system by \(F\). Then

\[
DF(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & -1 \end{bmatrix},
\]

so

\[
DF(X_-) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad DF(X_+) = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}.
\]

This means that \(X_-\) is a real sink and \(X_+\) is a saddle.

(b) If \(x = \pm 1\), then \(x' = 0\), so \(x(t)\) is constant. Therefore, the lines \(x = \pm 1\) are invariant. If \(y = 0\), then \(y' = 0\), so \(y(t)\) is constant. Therefore, the line \(y = 0\) is invariant. This implies that the stable curve for \(X_+\) is the line \(x = 1\), whereas its unstable curve is the interval \((-1, \infty)\) on the \(x\)-axis. The phase portrait looks like this:

Figure 3: Phase portrait of \(x' = x^2 - 1, y' = -y\).
5. **(20+10 points)** (a) Compute the flow $\phi_t(x_0, y_0)$ of the system

$$
x' = x - 4y^3
\quad y' = -y.
$$

(b) **(extra credit, 10 points)** Describe the stable $W^s(0)$ and unstable $W^u(0)$ curve associated with the equilibrium at the origin.

**Solution:** (a) Let us compute the solution $(x(t), y(t))$ with $x(0) = x_0$ and $y(0) = y_0$. The solution to the second equation is $y(t) = y_0 e^{-t}$. Substituting into the first equation, we obtain the following linear equation:

$$
x' - x = -4y_0^3 e^{-3t}.
$$

An integrating factor is $\mu(t) = e^{\int(-3t)} = e^{-t}$, so the general solution is

$$
x(t) = \frac{1}{\mu(t)} \left( \int \mu(t)(-4)y_0^3 e^{-3t} dt + C \right)
= e^t \left( \int (-4)y_0^3 e^{-4t} dt + C \right)
= e^t (y_0^3 e^{-4t} + C)
= y_0^3 e^{-3t} + Ce^t.
$$

Taking $t = 0$, we obtain $C = x_0 - y_0^3$. Therefore,

$$
x(t) = y_0^3 e^{-3t} + (x_0 - y_0^3)e^t.
$$

Therefore, the flow is

$$
\phi_t(x_0, y_0) = \left( y_0^3 e^{-3t} + (x_0 - y_0^3)e^t, y_0 e^{-t} \right).
$$

(b) A point $(x_0, y_0)$ is in $W^s(0)$ iff $\phi(x_0, y_0) \to (0, 0)$, as $t \to \infty$. This is the case iff $x_0 - y_0^3 = 0$. Similarly, $(x_0, y_0) \in W^u(0)$ iff $\phi(x_0, y_0) \to (0, 0)$, as $t \to -\infty$, which is the case iff $y_0 = 0$ (the $x$-axis). Therefore,

$$
W^s(0) = \{(x, y) : x = y^3\}, \quad W^u(0) = \{(x, y) : y = 0\}.
$$
6. (20 points) Consider the family of linear systems
\[
\begin{align*}
x' &= ax - y \\
y' &= x + ay,
\end{align*}
\]
where \(a\) is a parameter.

(a) As \(a\) varies over the real numbers, plot the curve in the trace-determinant plane corresponding to this family.

(b) Determine all values of \(a\) for which \(L(x, y) = x^2 + y^2\) is a strict Lyapunov function for the corresponding system.

Solution: (a) The trace and determinant are \(T = 2a, D = a^2 + 1\). The equation of the corresponding curve in the trace-determinant plane is thus
\[
D = \left(\frac{T}{2}\right)^2 + 1, \quad \text{or} \quad T^2 - 4D = -4.
\]
The curve is therefore a parabola, which can be obtained by shifting the critical parabola \(T^2 = 4D\) be one unit upward, as in Fig. 4.

![Figure 4: The parabola \(D = (T/2)^2 + 1\).](image)

(b) We have:
\[
\dot{L}(x, y) = 2xx' + 2yy' = 2x(ax - y) + 2y(x + ay) = 2a(x^2 + y^2).
\]
Thus \(\dot{L} < 0\) on \(\mathbb{R}^2 \setminus \{0\}\) iff \(a < 0\). Therefore, since \(L > 0\) everywhere except at the origin, \(L\) is a strict Lyapunov function for \(a < 0\).

Note that this is consistent with the fact that if \(a < 0\), then the corresponding system has a spiral sink at the origin.