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**Explain your work**
1. Consider the following differential equation on the real line:

\[ x' = x^3 - 2x^2 - 3x. \]

(a) Find the equilibria and identify their type.

(b) Draw the phase line.

(c) If \( x(t) \) is the solution satisfying \( x(0) = 2 \), what is the limit of \( x(t) \) as \( t \to \infty \)?

Solution:
2. Consider the following planar linear system:

\[ X' = AX, \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}. \]

(a) Find the eigenvalues and eigenvectors of \( A \).

(b) Find the matrix \( T \) that puts \( A \) in canonical form.

(c) Find the general solution of both \( X' = AX \) and \( Y' = BY \), where \( B = T^{-1}AT \).

(d) Sketch the phase portraits of both systems.

Solution:
3. Given are four planar linear systems, $X' = A_i X$, $i = 1, 2, 3, 4$. If

$A_1 = \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & 1 \\ -3 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 7 \\ 0 & -3 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1,000 & 1 \\ 10,000 & 2 \end{bmatrix},$

determine which systems are topologically conjugate to each other.

Solution:
4. Consider the planar system of differential equations in polar coordinates:

\[ r' = r(r - 1) \]
\[ \theta' = -1. \]

Sketch the phase portrait.

**Solution:**
5. Consider the following planar dynamical system:

\[
\begin{align*}
x' &= -y - x(x^2 + y^2) \\
y' &= 2x - y(x^2 + y^2).
\end{align*}
\]

(a) Show that the origin is the only equilibrium and describe the phase portrait of the linearized system. What does the linearized system tell us about the behavior of solutions of the nonlinear system near the origin?

(b) Let \( L(x, y) = 2x^2 + y^2 \). Compute \( \dot{L}(x, y) \).

(c) What can be said about the stability of the origin as an equilibrium of the given nonlinear system?

Solution:
6. Consider the following planar dynamical system:

\[ \begin{align*}
    x' &= x(x - 1) \\
    y' &= y(1 - y).
\end{align*} \]

(a) Find the equilibria and determine their type.

(b) Approximately sketch the phase portrait.

Solution: