Silver’s Theorem. Relative to the consistency of a cardinal \( \kappa \) that is \( \kappa^{++} \)-supercompact, it is consistent that there exists a measurable cardinal \( \kappa \) such that \( 2^\kappa = \kappa^{++} \).

A theorem of Kunen’s [J, pp. 450ff.] states that if the GCH fails at a measurable cardinal, then it is consistent that there exist \( \alpha \) many measurable cardinals, for each ordinal \( \alpha \). Thus some hypothesis stronger than measurability is necessary to obtain the consistency of the GCH failing at a measurable cardinal.

Furthermore, if the GCH fails at a measurable cardinal, then it must fail at many smaller inaccessible cardinals.

Silver originally proved his theorem using a backwards Easton support iteration of Cohen product forcing that adds \( \alpha^{++} \) many subsets to each inaccessible \( \alpha \) less than or equal to \( \kappa \). The purpose of this note is to give an alternate proof that avoids such a backwards Easton forcing in favor of a forwards Easton forcing that adds a \( \kappa \)-tree with \( \kappa^{++} \) many branches while preserving the measurability of \( \kappa \). Only this part of the proof is different from Silver’s; our use of supercompactness is essentially the same.

The forwards Easton forcing

For inaccessible cardinals \( \alpha \), set \( T_\alpha = \{ \alpha \} \times \alpha^{++} \). Let \( \alpha^* \) denote the least inaccessible cardinal greater than \( \alpha \), provided that there exists an inaccessible greater than \( \alpha \).

Fix an inaccessible cardinal \( \kappa \) and declare that \( p \in \mathbb{P}^\kappa \) iff \( p \) is a function and

1. the domain of \( p \) is an Easton set of inaccessible cardinals less than \( \kappa \).
2. if \( \alpha \) lies in the support of \( p \), then \( p_\alpha : T_\kappa \to T_\alpha \) and \( |p_\alpha| < \alpha^* \);
3. if \( \alpha < \beta \) lie in the support of \( p \), then \( \text{dom}(p_\alpha) \subseteq \text{dom}(p_\beta) \); and
4. (no M’s) if \( \alpha < \beta \) lie in the support of \( p \) and \( u, u' \in \text{dom}(p_\alpha) \) and if \( p_\beta(u) = p_\beta(u') \), then \( p_\alpha(u) = p_\alpha(u') \).

Declare that \( \bar{p} \geq p \) iff \( \text{sp}(\bar{p}) \subseteq \text{sp}(p) \) and \( \bar{p}_\alpha \subseteq p_\alpha \), for each \( \alpha \in \text{sp}(\bar{p}) \).

The motivation for this definition is that each \( u \) in \( T_\kappa \) should index a branch through a generic \( \kappa \)-tree, specifically that \( p_\alpha(u) = t \) commits \( t \) to lie in the branch indexed by \( u \).

In the constructions that follow, properties (1)–(3) will be uncontroversial; only (4) will require verification. Call objects satisfying properties (1)–(3) pseudo-conditions. Then to verify that a pseudo-condition is a condition, we must check that it has no M’s.

This is version 1.1.
If $p$ is a pseudo-condition, set $r(p) = \bigcup_{\alpha \in \text{sp}(p)} \text{rng}(p_\alpha)$ and define a relation $\rightarrow_{p,i}$ on $r(p)$ by

$$t \rightarrow_{p,i} t' \iff t \in T_\alpha \text{ and } t' \in T_\beta, \text{ for some } \alpha < \beta, \text{ and there exists a } u \in T_\kappa \text{ such that } p_\alpha(u) = t \text{ and } p_\beta(u) = t'. $$

That is, $t \rightarrow_{p,i} t'$ when $t \in T_\alpha$ and $t' \in T_\beta$, for some $\alpha < \beta$, and $t$ and $t'$ have a common preimage under $p$.

Let $\rightarrow_p$ be the transitive closure of $\rightarrow_{p,i}$. Then $t \rightarrow_p t'$ iff there exists a finite sequence $t_0, \ldots, t_n$ such that $t = t_0 \rightarrow_{p,i} \cdots \rightarrow_{p,i} t_n = t'$.

Clearly $\rightarrow_p$ is a well-founded partial ordering.

**Lemma.** Suppose that $p$ is a pseudo-condition. Then $\rightarrow_p$ is a tree ordering iff $p$ is a condition.

**Proof:** ($\Leftarrow$) Suppose that $\rightarrow_p$ is not a tree ordering. Then the $\rightarrow_p$-predecessors of some $t$ are not linearly ordered. Choose $\alpha$ to be least such that there exists $t \in T_\alpha$ with $t', t'' \rightarrow_p t$ and $t' \not\sim t''$ in $\rightarrow_p$. Then we may assume that $t', t'' \rightarrow_{p,i} t$. Say $t' \in T_\alpha'$ and $t'' \in T_\alpha''$ and $\alpha'' \leq \alpha'$. Let $u' \in T_\kappa$ be a common preimage under $p$ of $t$ and $t'$, and let $u''$ be a common preimage under $p$ of $t$ and $t''$. Now $u'' \in \text{dom}(p_{\alpha''})$ and $p_{\alpha''}(u'') \neq t'$, or else $u''$ is a common preimage under $p$ of both $t''$ and $t'$, contradicting that $t'' \not\rightarrow_p t'$. But then $p_\alpha(u') = t = p_\alpha(u'')$ and $p_{\alpha''}(u'') \neq t' = p_{\alpha''}(u')$ and so $p$ is not a condition.

($\Rightarrow$) Suppose that $\alpha < \beta$ lie in the support of $p$, that $u, u' \in \text{dom}(p_\alpha)$, and that $p_\beta(u) = p_\beta(u')$, but that $p_\alpha(u) \neq p_\alpha(u')$. Then $p_\alpha(u), p_\alpha(u') \rightarrow_p p_\beta(u) = p_\beta(u')$, but $p_\alpha(u)$ and $p_\alpha(u')$ are not $\rightarrow_p$-comparable. $\Box$

**Extension Lemma.** Suppose that $\bar{p} \in \mathbb{P}_\kappa$.

(a) If $\beta < \kappa$ is inaccessible, then there exists a condition $p$ extending $\bar{p}$ with $\beta \in \text{sp}(\bar{p})$ and $u \in \text{dom}(p_\beta)$.

(b) if $\beta \in \text{sp}(\bar{p})$ and $u \in T_\kappa$, then there exists a condition $p$ extending $\bar{p}$ such that $u \in \text{dom}(p_\beta)$ and $\text{sp}(p) = \text{sp}(\bar{p})$.

**Proof of (a):** Suppose that $\beta \notin \text{sp}(\bar{p})$. Set $U = \bigcup_{\alpha \in \text{sp}(\bar{p}) \cap \beta} \text{dom}(p_\alpha)$. For $u, u' \in U$, declare that $u \sim u'$ iff $\bar{p}_\alpha(u) = \bar{p}_\alpha(u')$, for all sufficiently large $\alpha \in \text{sp}(\bar{p}) \cap \beta$. Then $\sim$ is an equivalence relation on $U$. Now $|U| < \beta$, so there exists a function $f: U \rightarrow T_\beta$ such that $f(u) = f(u')$ iff $u \sim u'$. Let $p$ be identical with $\bar{p}$, except that $\beta \in \text{sp}(p)$ and $p_\beta = f$. Then $p$ is a pseudo-condition.

Suppose that $\delta < \gamma$ lie in $\text{sp}(p)$, that $u, u' \in \text{dom}(p_\delta)$, and that $p_\gamma(u) = p_\gamma(u')$. We must argue that $p_\delta(u) = p_\delta(u')$. Since $\bar{p}$ is a condition, we may assume that $\delta = \beta$ or $\gamma = \beta$.

If $\delta = \beta$, then $u, u' \in U$, hence $u, u' \in \text{dom}(p_\alpha)$, for all sufficiently large $\alpha \in \text{sp}(\bar{p}) \cap \beta$. Furthermore, $\bar{p}_\alpha(u) = \bar{p}_\alpha(u')$, for all such $\alpha$, because $\bar{p}_\gamma(u) = \bar{p}_\gamma(u')$. Hence $u \sim u'$ and $p_\beta(u) = p_\beta(u')$.

If $\gamma = \beta$, then $p_\beta(u) = p_\beta(u')$. Hence $u \sim u'$. It follows that $\bar{p}_\alpha(u) = \bar{p}_\alpha(u')$, for some $\alpha$ such that $\delta \leq \alpha < \beta$, and so that $p_\delta(u) = p_\delta(u')$.

**Proof of (b):** Fix $u \in T_\kappa$ and suppose that $u \notin \text{dom}(\bar{p}_\beta)$. 

Note first that we may assume that $\text{dom}(\bar{p}_\alpha) \neq \emptyset$, for some $\alpha \in \text{sp}(\bar{p})$. Otherwise, we could choose $t_\alpha \in T_\alpha$, for each $\alpha \in \text{sp}(\bar{p})$, and extend $\bar{p}$ to obtain $p$ with $\text{dom}(p_\alpha) = \{u\}$ and $p_\alpha(u) = t_\alpha$, for each $\alpha \in \text{sp}(\bar{p}) = \text{sp}(p)$.

In fact, we may assume that $\text{dom}(\bar{p}_\alpha) \neq \emptyset$, for all $\alpha \in \text{sp}(\bar{p})$. Otherwise, we could choose $\alpha$ to be least such that $\text{dom}(\bar{p}_\alpha) \neq \emptyset$, choose any $u' \in \text{dom}(\bar{p}_\alpha)$ and any $t_\alpha \in T_{\gamma}$, for $\gamma \in \text{sp}(\bar{p}) \cap \alpha$, and let $p$ extending $\bar{p}$ be identical with $p$, except that for $\gamma \in \text{sp}(\bar{p}) \cap \alpha$, $\text{dom}(p_\gamma) = \{u'\}$ and $p_\gamma(u') = t_\alpha$.

It follows that we may further assume that $u \in \text{dom}(\bar{p}_\alpha)$, for some $\alpha \in \text{sp}(\bar{p})$. If not, we could choose any $u' \in \text{dom}(\bar{p}_\alpha)$, where $\alpha$ is least in $\text{sp}(\bar{p})$, and let $p$ extending $\bar{p}$ be identical with $\bar{p}$, except that $\text{dom}(p_\alpha) = \text{dom}(\bar{p}_\alpha) \cup \{u\}$ and $p_\alpha(u) = \bar{p}_\gamma(u')$, for all $\gamma \in \text{sp}(\bar{p})$.

Let us now fix $\alpha \in \text{sp}(\bar{p})$ to be least such that $u \in \text{dom}(\bar{p}_\alpha)$. Note that we may assume that $\bar{p}_\alpha(u)$ is $\rightarrow_\bar{p}$-minimal. Indeed, if there exists $\bar{\alpha} < \alpha$ such that $\bar{p}_\alpha(u)$ has a $\rightarrow_\bar{p}$-predecessor in $T_{\bar{\alpha}}$, then, letting $p$ be identical with $\bar{p}$, except that $u \in \text{dom}(p_\alpha)$ and $p_\alpha(u)$ is the $t \in T_{\gamma}$ such that $t \rightarrow_\bar{p} \bar{p}_\alpha(u)$, for each $\gamma \in \text{sp}(\bar{p}) \cap [\bar{\alpha}, \alpha)$, we obtain a pseudo-condition $p$ such that $\rightarrow_\bar{p}$ is identical with $\rightarrow_\bar{p}$. It follows that $p$ is a condition.

So let us suppose that $\bar{p}_\alpha(u)$ is $\rightarrow_\bar{p}$-minimal. Choose any $u' \in \text{dom}(\bar{p}_\beta)$ and let $p$ be identical with $\bar{p}$, except that $p_\gamma(u) = \bar{p}_\gamma(u')$, for $\gamma \in \text{sp}(\bar{p}) \cap [\beta, \alpha)$. Then $p$ is a pseudo-condition and

$$\rightarrow_\bar{p} = \rightarrow_{\bar{p}} \cup \left\{ (t', t) : \bar{p}_\alpha(u) \vdash_{\bar{p}} t \text{ and } t' \vdash_{\bar{p}} \bar{p}_\gamma(u') \text{ for some } \gamma \in \text{sp}(\bar{p}) \cap [\beta, \alpha) \right\}$$

Using that $\rightarrow_{\bar{p}}$ is a tree ordering, it can be seen that $\rightarrow_\bar{p}$ is as well. Hence $p$ is as required. $\square$

Set $T = \bigcup_{\alpha < \kappa} T_\alpha$ and suppose that $G$ is a filter on $\mathbb{P}^\kappa$. Set $\rightarrow_G = \bigcup_{p \in G} \rightarrow_p$. Then $\rightarrow_G$ is a tree ordering on $T$.

Below we shall argue that $\mathbb{P}^\kappa$ is cardinal preserving and calculate cardinal exponents in $\mathbb{P}^\kappa$ generic extensions. The following proposition simply remarks that $\mathbb{P}^\kappa$ conforms to its motivation.

**Tree Lemma.** Suppose that $\kappa$ is an inaccessible limit of inaccessibles and that $G$ is $\mathbb{P}$ generic over $V$. Then $(T, \rightarrow_G)$ is a $\kappa$-tree with $\kappa^{++}$ many branches.

**Proof:** Using that $G$ is generic and that $\kappa$ is a limit of inaccessibles, it is easy to see that $(T, \rightarrow_G)$ has height $\kappa$. Using that $|T_\alpha| = \kappa^{++} < \kappa$, for inaccessible $\alpha < \kappa$, it follows that $(T, \rightarrow_G)$ is a $\kappa$-tree.

For $u \in T_\kappa$, set

$$b_u = \{ p_\alpha(u) : p \in G \text{ and } \alpha \in \text{sp}(p) \text{ and } u \in \text{dom}(p_\alpha) \}.$$

Suppose that $u \neq u'$ lie in $T_\kappa$ and that $u, u' \in d(p) = \bigcup_{\alpha \in \text{sp}(p)} \text{dom}(\bar{p}_\alpha)$, for some $\bar{p} \in \mathbb{P}^\kappa$. Choose an inaccessible $\beta$ such that $\sup(\text{sp}(\bar{p})) < \beta < \kappa$. Now $|d(\bar{p})| < \sup(\text{sp}(\bar{p}))^+ \leq \beta$, so there exists a one-to-one function $f : d(\bar{p}) \to T_\beta$. Let $p$ be identical with $\bar{p}$, except that $\text{sp}(p) = \text{sp}(\bar{p}) \cup \{\beta\}$ and $p_\beta = f$. Then $p$ is a condition extending $\bar{p}$ and $p \Vdash b_u \neq b_{u'}$. $\square$
ANTICHAIN LEMMA. Suppose that $\kappa$ is inaccessible, but not the least inaccessible cardinal. Set $\lambda = \sup\{ \mu^* : \mu < \kappa \text{ is inaccessible}\}$. Then $\mathbb{P}^\kappa$ has antichains of cardinality at most $\lambda^{<\kappa}$.

If $\kappa$ is the least inaccessible, then $\mathbb{P}^\kappa = \{\emptyset\}$.

If $\kappa$ is inaccessible and is greater than the least inaccessible cardinal, then either $\lambda = \kappa$, or $\lambda < \kappa$ is a singular strong limit cardinal. In the former case $\lambda^{<\lambda} = \kappa$; in the latter, $2^\lambda \leq \lambda^{cf(\lambda)}$, so $\lambda^{<\lambda} = \lambda^\lambda = 2^\lambda = \lambda^{cf(\lambda)}$.

PROOF: Suppose that $A \subseteq \mathbb{P}^\kappa$ has cardinality $(\lambda^{<\lambda})^+$. We must see that $A$ is not an antichain. Noting that $\lambda$ has $\lambda^{<\lambda}$ many Easton subsets, we may assume that there exists a fixed $D \subseteq \lambda$ such that $sp(p) = D$, for all $p \in A$. As previously, set $d(p) = \bigcup_{\alpha \in D} \text{dom}(p_\alpha)$. We may assume that $\mathcal{F} = \{d(p) : p \in A\}$ forms a $\Delta$-system.

(If $\lambda = \kappa$, then without loss of generality, $\mathcal{F}$ is a family of $\kappa^+$ many sets, each of cardinality less than $\kappa$; if $\lambda < \kappa$, then, without loss of generality, $\mathcal{F}$ is a family of $(2^\lambda)^+$ many sets, each of cardinality less than $\lambda^+$.), Say the root of $\mathcal{F}$ is $r$. We may assume that $\text{rng}(p_\alpha) = \text{rng}(p'_\alpha)$, for all $p, p' \in A$ and all $\alpha \in D$. Finally, if $p, p' \in A$, we may assume that there exists a bijection $\varepsilon : d(p) \to d(p')$ such that

- $e \restriction r = \text{id} \restriction r$;
- $u \in \text{dom}(p_\alpha)$ iff $e(u) \in \text{dom}(p'_\alpha)$, for all $\alpha \in D$ and $u \in d(p)$; and
- $p_\alpha(u) = p'_\alpha(e(u))$, for all $\alpha \in D$ and $u \in \text{dom}(p_\alpha)$.

(Consider isomorphism types of $p \in A$ in a language with function symbols for each $p_\alpha$ and constant symbols for each element of $r$ and of $\bigcup_{\alpha \in D} \text{rng}(p_\alpha)$.) Call such an $e$ an isomorphism from $p$ to $p'$.

Suppose that $p, p' \in A$. Let $sp(q) = D$ and set $q_\alpha = p_\alpha \cup p'_\alpha$, for $\alpha \in D$. We maintain that $q$ is a condition extending both $p$ and $p'$. To see that when $\alpha \in D$, the set $q_\alpha$ is a function, let $e$ be an isomorphism from $p$ to $p'$. If $u \in \text{dom}(p_\alpha) \cap \text{dom}(p'_\alpha)$, then $u \in r$ and so $p_\alpha(u) = p'_\alpha(e(u)) = p'_\alpha(u)$. Thus $q$ is a pseudo-condition.

Suppose now that $\alpha < \gamma$ lie in $D$, that $u, u' \in \text{dom}(q_\alpha)$, and that $q_\alpha(u) = q_\alpha(u')$. We must see that $q_\alpha(u) = q_\alpha(u')$. We may assume that $u \in \text{dom}(p_\gamma)$ and $u' \in \text{dom}(p'_\gamma)$. Let $e$ be an isomorphism from $p$ to $p'$. Then $p_\gamma(u) = p'_\gamma(u')$, so $p'_\gamma(e(u)) = p'_\gamma(u')$. But then $p'_\alpha(e(u)) = p'_\alpha(u')$. Hence $p_\alpha(u) = p'_\alpha(u')$. $\Box$

Suppose that $\mu < \kappa$ and $\mu$ is inaccessible. Set

$$
\mathbb{P}^\mu_\mu = \{ p \in \mathbb{P}^\kappa : sp(p) \cap \mu = \emptyset \}.
$$

Then $\mathbb{P}^\kappa_\mu$ is $<\mu^{\kappa}$-closed.

FACTOR LEMMA. Suppose that $\mu < \kappa$ are inaccessible. Then $\mathbb{P}^\kappa$ is equivalent to the product $\mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu$.

PROOF: It suffices to define a function

$$
e : \{ p \in \mathbb{P} : \mu \in sp(p) \} \to \mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu
$$

such that

- $\bar{p} \geq p$ iff $\bar{e}(p) \geq e(p)$, and
- the range of $e$ is dense in $\mathbb{P}^\mu \times \mathbb{P}^\kappa_\mu$. 
Suppose that \( p \in \mathcal{P}^\kappa \) and that \( \mu \in \text{sp}(p) \). Set \( e(p) = (q, r) \), where \( r = p \upharpoonright [\mu, \kappa) \) and \( q \) is defined as follows: Set \( \text{sp}(q) = \text{sp}(p) \cap \mu \). For \( \alpha \in \text{sp}(q) \), set \( \text{dom}(q_\alpha) = p^-\alpha \land \text{dom}(p_\alpha) \), which has cardinality less than \( \alpha^* \). If \( t \in \text{dom}(q_\alpha) \), set \( q_\alpha(t) = p_\alpha(u) \), where \( u \in p_\alpha^{-1}(t) \cap \text{dom}(p_\alpha) \). A key observation is that this definition of \( q_\alpha(t) \) does not depend on our choice of \( u \). Indeed, if \( u, u' \in \text{dom}(p_\alpha) \) and \( p_\alpha(u) = p_\alpha(u') = t \), then \( p_\alpha(u) = p_\alpha(u') \). This completes the definition of \( e \).

Assume first that \( \bar{p} \supseteq p \). Set \( e(\bar{p}) = (\bar{q}, \bar{r}) \) and \( e(p) = (q, r) \). We must see that \((\bar{q}, \bar{r}) \supseteq (q, r)\). Certainly \( \bar{r} \supseteq r \). Suppose, then, that \( t \in \text{dom}(\bar{q}_\alpha) \). Say \( \bar{p}_\alpha(u) = t \). Then \( p_\alpha(u) = t \), so \( q_\alpha(t) = p_\alpha(u) = p_\alpha(u) = q_\alpha(t) \). Hence \( \bar{q} \supseteq q \), as well.

Conversely, suppose that \((\bar{q}, \bar{r}) \supseteq (q, r)\), where \( e(\bar{p}) = (\bar{q}, \bar{r}) \) and \( e(p) = (q, r) \). We must see that \( \bar{p}_\alpha \subseteq p_\alpha \), for all \( \alpha \in \text{sp}(\bar{p}) \). This is clear if \( \alpha \geq \mu \), since \( \bar{r} \supseteq r \). If \( \alpha < \mu \) and \( u \in \text{dom}(\bar{p}_\alpha) \), set \( t = \bar{p}_\alpha(u) \). Then \( t \in \text{dom}(\bar{q}_\alpha) \) and \( \bar{q}_\alpha(t) = \bar{p}_\alpha(u) \). Since \( \bar{p}_\alpha = \bar{r}_\alpha \subseteq r_\alpha = p_\alpha \), also \( q_\alpha(t) = p_\alpha(u) \). And \( q_\alpha(t) = q_\alpha(u) \), since \( \bar{q}_\alpha \subseteq q_\alpha \).

Finally, to see that the range of \( e \) is dense in \( \mathcal{P}^\mu \times \mathcal{P}^\mu \), note first that the collection of pairs \((q, r)\) such that \( q \in \text{sp}(r) \) and \( \text{dom}(q_\alpha) \subseteq \text{rng}(r_\alpha) \), for all \( \alpha \in \text{sp}(q) \), is dense in \( \mathcal{P}^\mu \times \mathcal{P}^\mu \). This uses the Extension Lemma, that \(|d(q)| < \mu \), and that \( \mathcal{P}^\mu \) is \(<\mu^*\)-closed.

If \( (q, r) \) is such a pair, then define \( p \in \mathcal{P}^\kappa \) with \( \text{sp}(p) = \text{sp}(q) \cup \text{sp}(r) \) as follows: If \( \alpha \in \text{sp}(r) \), set \( p_\alpha = r_\alpha \). If \( \alpha \in \text{sp}(q) \), for each \( t \in \text{dom}(q_\alpha) \), choose \( u_t \in \text{dom}(r_\alpha) \) such that \( r_\alpha(u_t) = t \). Set \( \text{dom}(p_\alpha) = \{ u_t : t \in \text{dom}(q_\alpha) \} \), which has cardinality less than \( \alpha^* \), and set \( p_\alpha(u_t) = q_\alpha(t) \). Then \( e(p) = (q, r) \). \( \square \)

**Cardinal Preservation Lemma.** Assume the GCH in the ground model. If \( \kappa \) is inaccessible and \( \alpha \) is a regular cardinal, then in a \( \mathcal{P}^\kappa \)-generic extension the range of each \( \alpha \)-sequence is covered by a ground model set of ground model cardinality \( \alpha \). Consequently, \( \mathcal{P}^\kappa \) is cardinal preserving.

**Proof:** Fix \( \alpha \). Since \( \mathcal{P}^\kappa \) satisfies the \( <\kappa^+\)-chain condition, the lemma is clear if \( \alpha \geq \kappa \).

Suppose that \( \alpha < \kappa \).

**Case 1.** There exists a largest inaccessible \( \mu \leq \alpha \). Then \( \mathcal{P}^\kappa \) is equivalent to \( \mathcal{P}^\mu \times \mathcal{P}^\mu \).

Now \( \mathcal{P}^\mu \) is \( \leq \alpha \)-closed, because \( \alpha < \mu^* \), and \( \mathcal{P}^\mu \) has antichains of size at most \( \mu \). Our claim follows.

**Case 2.** There does not exist a largest inaccessible less than or equal to \( \alpha \). Set \( \lambda = \sup\{ \nu < \alpha : \nu \text{ is inaccessible} \} = \{ \nu^* < \alpha : \nu \text{ is inaccessible} \} \). Then \( \lambda \) is singular or \( \lambda = 0 \), so \( \lambda < \alpha \). Let \( \mu \) be the least inaccessible greater than \( \alpha \). Now \( \mathcal{P}^\kappa \) is equivalent to \( \mathcal{P}^\mu \times \mathcal{P}^\mu \). And \( \mathcal{P}^\mu \) is \( \leq \alpha \)-closed; and if \( \mathcal{P}^\mu \) is non-trivial, then \( \mathcal{P}^\mu \) has antichains of size at most \( \lambda^{<\lambda} = \lambda^+ \leq \alpha \). \( \square \)

**Cardinal Exponentiation Lemma.** Assume the GCH in the ground model \( V \). For inaccessible \( \mu \), set

\[ \lambda_\mu = \sup\{ \nu^* : \nu < \mu \text{ is inaccessible} \} \]

Suppose that \( \kappa \) is inaccessible, that \( G \) is \( \mathcal{P}^\kappa \)-generic, and that \( \alpha \) is an infinite cardinal greater than the least inaccessible. Then \( 2^\alpha = \mu^{++} \) in \( V[G] \), if \( \alpha = \mu \) or if \( \lambda_\mu < \alpha < \mu \), for some inaccessible \( \mu \leq \kappa \). Otherwise \( 2^\alpha = \alpha^+ \).

**Proof:** If \( \alpha > \kappa \), then \( 2^\alpha = \alpha^+ \), since \( |\mathcal{P}^\kappa| = \kappa^{++} \leq \alpha^+ \).

For infinite cardinals \( \alpha \leq \kappa \), proceed by induction on \( \alpha \).

If \( \alpha \) is less than or equal to the least inaccessible cardinal \( \mu_0 \), then \( 2^\alpha = \alpha^+ \) in \( V[G] \), since \( \mathcal{P}^{\mu_0} \) is trivial and \( \mathcal{P}^\kappa \) is \( \leq \alpha \)-closed.
If \( \alpha \) lies in the interval \((\mu_0, \kappa)\) and is not inaccessible and does not lie in any interval \((\lambda_\mu, \mu)\), for \( \mu \leq \kappa \), then there are two cases to consider, namely, that \( \alpha \) is singular and that there exists a largest inaccessible less than \( \alpha \).

If \( \alpha \) is singular, then by induction \( \alpha \) is a strong limit cardinal in \( V[G] \). It follows by the covering claim of the previous lemma that \( 2^\alpha = \alpha^+ \) in \( V[G] \).

And if there exists a largest inaccessible \( \mu \) less than \( \alpha \), using that \( |\mathbb{P}^\mu| \leq \mu^{++} \leq \alpha^+ \) and that \( \mathbb{P}^\mu \) is \( \leq \alpha \)-closed, it follows that \( 2^\alpha = \alpha^+ \) in \( V[G] \).

Now suppose that \( \alpha \) lies in an interval \((\lambda_\mu, \mu)\), for some inaccessible \( \mu < \kappa \). It suffices to see that \( \mathbb{P}^\mu \) adds \( \mu^{++} \) distinct functions from \( \lambda_\mu^+ \) into \( \mathcal{P}(\lambda_\mu) \) because \( (2^\lambda_\mu)^{\lambda_\mu^+} = 2^{\lambda_\mu^+} \leq 2^\alpha \). (Conversely, \( 2^\alpha \leq \mu^{++} \) because \( |\mathbb{P}^\mu| \leq \mu^{++} \) and \( \mathbb{P}^\mu \) is \( \leq \alpha \)-closed.) Begin by noting that if \( p \in \mathbb{P}^\mu \), then \( \left| \bigcup_{i \in \text{sp}(p)} \text{dom}(p_i) \right| \leq \lambda_\mu \). Consequently, given any two disjoint \( \lambda_\mu^+ \)-sequences \( \vec{t} \) and \( \vec{t}' \) from \( T_\mu \), there exists a dense collection of \( p \) such that \( p_\gamma \) assigns different elements of \( T_\gamma \) to the \( i \)th element of \( \vec{t} \) and the \( i \)th element of \( \vec{t}' \), for some \( \gamma < \lambda_\mu \) and some \( i < \lambda_\mu^+ \). Thus if \( G \) is generic over \( \mathbb{P}^\mu \), then the following two sequences are distinct:

\[
\left\{ p_\gamma(\vec{t}_i) : \gamma < \lambda_\mu \text{ is inaccessible} \right\} : i < \lambda_\mu^+ \\
\text{and} \\
\left\{ p_\gamma(\vec{t}'_i) : \gamma < \lambda_\mu \text{ is inaccessible} \right\} : i < \lambda_\mu^+.
\]

Finally, if \( \alpha \leq \kappa \) is inaccessible, it suffices to see that forcing with \( \mathbb{P}^\alpha \) adds \( \alpha^{++} \) many functions from \( \alpha \) into \( \mathcal{P}(\alpha) \). For this, note that if \( p \in \mathbb{P}^\alpha \), then \( \left| \bigcup_{i \in \text{sp}(p)} \text{dom}(p_i) \right| < \alpha \) and proceed as in the previous case. \( \square \)

**Proof of Silver’s theorem**

We are now prepared to finish the proof of Silver’s theorem. Suppose that \( \kappa \) is \( \lambda \)-supercompact, where \( \lambda = \kappa^{++} \). Suppose that \( j : V \rightarrow M \) is elementary, where \( \kappa \) is the critical point of \( j \) and \( \lambda M \subseteq M \). Set \( \mathbb{P} = \mathbb{P}^\kappa \).

Since \( \mathbb{P} \subseteq \mathcal{H}_\lambda \) and \( |\mathbb{P}| = \lambda \), we have that \( \mathbb{P} \in M \). Furthermore, \( j(\mathbb{P})^\kappa = \mathbb{P} \). Set \( \mathbb{Q} = \left( j(\mathbb{P})^{j(\kappa)} \right)^M \). Then “\( \mathbb{P} \times \mathbb{Q} \) is equivalent to \( j(\mathbb{P}) \)” holds in \( M \), hence in \( V \). Let \( e : \{ p \in j(\mathbb{P}) : \kappa \in \text{sp}(p) \} \rightarrow \mathbb{P} \times \mathbb{Q} \) be as in the proof of the Factor Lemma. Because “\( \mathbb{Q} \) is \( \leq \lambda \)-closed” holds in \( M \) and \( \lambda M \subseteq M \), we have that \( \mathbb{Q} \) is, in fact, \( \leq \lambda \)-closed in \( V \).

Define the master condition \( \dot{\kappa} \in \mathbb{Q} \) as follows: Set \( \dot{\kappa}(\dot{\kappa}) = \{ \kappa \} \) and declare that \( \dot{\kappa}(\dot{\kappa}(t)) = t \), for each \( t \in T_\kappa \). Note that \( \dot{\kappa} \in \mathbb{Q} \), using that \( |j| T_\kappa| = \lambda \), hence \( j \upharpoonright T_\kappa \in M \).

Note also that if \( p \in \mathbb{P} \), then \( \text{sp}(j(p)) = \text{sp}(p) \) and \( j(p)_\alpha = p_\alpha \circ \dot{\kappa} \), for all \( \alpha \in \text{sp}(p) \). That is, in the notation of the Factor Lemma, \( e(j(p)) = (p, \dot{\kappa}) \).

Suppose that \( G \) is \( \mathbb{P} \) generic over \( V \). Choose \( H \) to be \( \mathbb{Q} \) generic over \( V[G] \) with \( \dot{\kappa} \in H \). Set

\[
K = \left\{ r \in j(\mathbb{P}) : r \geq r', \text{ for some } r' \text{ such that } u \in \text{sp}(r') \text{ and } e(r') \in G \times H \right\}.
\]
Then $K$ is $j(\mathbb{P})$ generic over $V$, hence over $M$. Now

\[ p \in G \iff (p, q) \in G \times H \iff e(j(p)) \in G \times H \iff j(p) \in K. \]

It follows that $j$ extends to an elementary $\hat{j}: V[G] \to M[K]$. (Set $\hat{j}^{V[G]}(\hat{x}) = j(\hat{x})^{V[G]}$)

and note that

\[
V[G] \models \varphi(\hat{x}) \Rightarrow V \models p \Vdash \varphi(\hat{x}), \text{ for some } p \in G \\
\Rightarrow M \models j(p) \Vdash j(\varphi(\hat{x})), \text{ for some } p \in G \\
\Rightarrow M[K] \models \varphi(j(\hat{x})).
\]

Working in $V[K]$, define an ultrafilter $U$ on $\kappa$ by

\[
X \in U \iff \kappa \in j(X).
\]

Then $U$ is a $<\kappa$-complete normal non-principal ultrafilter. But $|U| = \kappa^++ = \lambda$ and $Q$ is $\leq \lambda$-closed, so $U \in V[G]$.

Hence $\kappa$ is measurable and $2^\kappa = \kappa^+$ in $V[G]$. \qed