PHOTON DISPERSION IN CAUSAL SETS: THE FEYNMAN PATH SUM APPROACH

PAUL CRACIUNOIU, KATE ISAACS, BRUCE LANGDON, MOLLY MOSCOE, ANNA VU AND SLOBODAN N. SIMIĆ

Abstract. This paper is a report on the results of a student research project conducted at San José State University in the Spring of 2008. The goal of the project was to explore the possibility that, in the context of causal sets, the speed of a photon is a function of its energy. This question was motivated by the NASA launch of the GLAST satellite and was suggested by our NASA sponsor, Dr. Jeffrey Scargle. We used the Feynman path sum approach to compute the probability amplitudes of various transitions in a causal set. We showed that it is possible to express the continuous data using solely the combinatorial information given by the causal set. The results of our MATLAB simulations were promising although too preliminary to draw definite conclusions.

1. Introduction

This paper is an outcome of the student research project with the same name run by the Center for Applied Mathematics, Computation, and Statistics (CAMCOS) at San José State University in the Spring of 2008. The project was sponsored by Dr. Jeffrey Scargle from the NASA-Ames Research Center. The goal of the project and this paper was to explore the possibility that the speed of a photon is a function of its energy. This was done in the context of causal set theory, using the Feynman path sum approach for computing the probability amplitudes. The question was motivated by the NASA launch of the GLAST satellite (now known as the Fermi Gamma-ray Space Telescope, see [14]), which will detect high-energy photons and may be able to observe possible deviations of the speed of light from its usual value.

Causal set theory is one of the competing theories of quantum gravity whose basic postulate is that spacetime is fundamentally discrete. Spacetime is thus just a discrete partially ordered set satisfying a local finiteness condition (see Section 2 for details). With this in mind, the goal of the project can be stated as follows. Consider a path γ in a causal set corresponding to a trajectory of a photon from a light source to a detector. In

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classical physics, one assigns to each path γ traveled by a particle a quantity called the action $S[\gamma]$ (see Section 4), which is just the integral of the kinetic minus potential energy. In quantum mechanics, the Feynman path integral (see Section 3), which is essentially the "integral" of $\exp(iS[\gamma]/h)$ over all possible paths γ traveled by a particle, gives a way of computing the probability amplitude of a transition in a quantum system. We use this approach in the context of causal sets with the following goal in mind:

Goal. *Assign to each path γ physically meaningful values of the action $S[\gamma]$ and use the Feynman integral (i.e., sum) approach to compute the probabilities that the speed of the photon is a function of its energy E.*

The paper is organized as follows. In Section 2, we review the basics of causal set theory and describe the context for our work. Section 3 contains a brief outline of the Feynman path integral approach to quantum mechanics and how we modified it for use in causal sets. In Section 4 we define the action and introduce the concept of a causality matrix. Section 5 discusses our main results; ideas and directions for future research are in Section 6.

Acknowledgments. We would like to thank Dr. Jeffrey Scargle for proposing this topic, his time, patience, and constant help and encouragement. Many thanks also to the director of CAMCOS, Professor Tim Hsu, for keeping us focused and helping with the final presentation.

In the process of writing this article a related work [11] has come to our attention.

2. Causal Set Theory

The causal set theory (or program) is one of several approaches to quantum gravity. Quantum gravity is the field of theoretical physics attempting to unify quantum mechanics (the theory that describes three fundamental forces – electromagnetism, weak interaction, and strong interaction), with general relativity, the theory of the fourth fundamental force, gravity. The ultimate goal of quantum gravity is to find a "theory of everything" (TOE). A discussion of quantum gravity is beyond the scope of this paper; for more information, the reader is referred to [1]. The founder and main proponent of causal set theory is Rafael Sorkin [2].

The basic premise of causal set theory is that spacetime is fundamentally *discrete*. This premise is based on a result of David Malament [13], which states that if f is a map between two past and future distinguishing spacetimes which preserves their causal structure, then the map is a conformal isomorphism (that is, it is a smooth bijection that preserves angles, though not necessarily distances).

Formally speaking, a causal set (or causet) is a set C equipped with a relation $\prec$ with the following properties:

Anti-reflexivity: $x \not\prec x$, for all $x \in C$;
Transitivity: if $x \prec y$ and $y \prec z$, then $x \prec z$;
Local finiteness: for all $x, z \in C$, the set

$$\{y \in C : x \prec y \quad \text{and} \quad y \prec z\}$$

is finite.

That is, a causal set is just a partially ordered set satisfying the local finiteness condition.

A convenient way of representing a causal set $(C, \prec)$ is by means of a directed graph, visualized via a Hasse diagrams. In a Hasse diagram each dot represents an element of a causal set and only the relations not implied by transitivity are drawn as edges (i.e., if $x \prec y$ and $y \prec z$, then one does not draw an edge from x to z). See Figure 1.

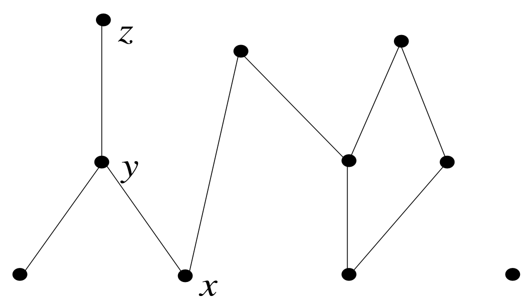

FIGURE 1. The Hasse diagram of a causal set.

Recall that to every graph G with vertices $V = \{v_1, \ldots, v_n\}$ and edges E one can associate a $n \times n$ matrix $A_G = [a_{ij}]$ called the adjacency matrix of G, where $a_{ij} = 1$ if there is an edge from v_i to v_j, otherwise $a_{ij} = 0$. We used this idea to define the concept of a causality matrix (see 4).

It is natural to ask how such a relatively simple structure as a causal set can be related to a Lorentzian manifold[1] used to represent spacetime in special relativity, i.e., how a causal set C can be *embedded* into a Lorentzian manifold L. An embedding is assumed to be a map $f : C \to L$ such that if $x \prec y$, then $f(x)$ would be in the causal past of $f(y)$. The embedding f is called faithful if for every subset $S \subset L$, the number of elements in the preimage $f^{-1}(S)$ is proportional to the volume of S. If such a faithful embedding exists, C is called manifold-like. The *hauptvermutung* (or fundamental conjecture) of causal set theory is that a causal set cannot be faithfully embedded into two spacetimes which are not similar on large scales (a term hard to define precisely).

[1]A Lorentzian manifold is a smooth manifold equipped with a pseudo-Riemannian metric with signature $(1, n-1)$, where n is the dimension of the manifold.

We remark that not every causal set can be embedded into a Lorentzian manifold; see [10] for an example and a more detailed discussion of this question.

It has been shown that given a causal set $(C, \prec)$ and volume information, it is possible to calculate the dimension, topology, differentiable structure, and metric of the corresponding Lorentzian manifold L (see [6, 9]).

A link in a causal set $(C, \prec)$ is a pair of elements (x, y) such that $x \prec y$ but there is no $z \in C$ such that $x \prec z$ and $z \prec y$. A chain is a sequence $(x_0, x_1, \ldots, x_n)$ of elements of C such that $x_i \prec x_{i+1}$, for all $0 \leq i \leq n-1$. The length of a chain $(x_0, x_1, \ldots, x_n)$ is defined to be n (the number of relations used).

A broader discussion of causal set theory is beyond the scope of this paper (and was beyond the scope of our project). For more details, the reader is referred to [3, 15, 10].

Given a (faithfully embeddable) causal set $(C, \prec)$, we focus on the following questions:

(a) Can continuous physical properties such as coordinate time, proper time, distance, and velocity be recovered using only the combinatorics of a causal set?
(b) How does one adopt the Feynman path integral approach for computing probability amplitudes of various quantum mechanical transitions in C?

Notation and terminology. For background on special relativity, the reader is referred to Feynman's lectures [8].

Given a causal set $(C, \prec)$ and $x, y \in C$ with $x \prec y$, the set

$$[x, y] = \{z \in C : x \prec z \prec y\}$$

is called the interval between x and y.

We will denote the cardinality of a set S by $|S|$.

Recall that if $f : C \to L$ is a faithful embedding of a causal set $(C, \prec)$ into a Lorentzian manifold L and $x \prec y$, then (see [10] for more details) $|[x, y]|$ approximately equals the volume (in L) of the intersection of the future light cone on $f(x)$ and the past light cone of $f(y)$.

We now introduce the light-cone coordinates in a 1+1 dimensional spacetime $\mathbb{R}^2$. Suppose (x, t) are the standard coordinates of a point p in a 1+1 dimensional spacetime $\mathbb{R}^2$. Set

$$x_+ = x + t, \quad x_- = -x + t.$$

Then (x_+, x_-) are called the light-cone coordinates of p. Observe that p is in the forward light cone of $(0, 0)$ if and only if (x_+, x_-) is in the first quadrant in the light-cone coordinate system. It is easy to see that

$$x = \frac{x_+ - x_-}{2}, \qquad t = \frac{x_+ + x_-}{2}.$$

3. The Feynman integral and Path Sums

Our approach to computing the probability amplitudes of paths in a causal set is based on the Feynman path integral. This subject is too vast to be reviewed here, so we only give a brief, informal discussion that suffices for our purposes. We follow [12].

Suppose that a particle of mass m is at position x_0 on the x-axis at time $t = 0$ and that it has potential energy $V(x_0)$. The particle moves to position x at some future time $t > 0$. According to the laws of quantum mechanics, this move is not deterministic but has a certain probability density $p(x,t)$ of occurring. This probability density is given by $p(x,t) = |K(x_0,x,t)|^2$, where K is a complex number called the probability amplitude. As a function it satisfies the Schrödinger equation

$$ih\frac{\partial K}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2 K}{\partial x^2} + V(x)K, \tag{1}$$

where h is Planck's constant divided by 2π. Since the particle has position x_0 at time zero, K has to satisfy the initial condition

$$K(x_0,x,0) = \delta(x - x_0), \tag{2}$$

where δ is the "delta function" defined by

$$\delta(y) = \begin{cases} 0, & \text{if} \quad y \neq 0 \\ \infty, & \text{if} \quad y = 0. \end{cases}$$

More precisely put, δ is a generalized function or a distribution satisfying

$$\int_{-\infty}^{\infty} \delta(x-y)f(y)\,dy = f(x),$$

for every continuous function $f : \mathbb{R} \to \mathbb{R}$.

Suppose we are given a function ψ_0 and would like to construct the solution $\psi(x,t)$ to the Schrödinger equation (1) satisfying the initial condition $\psi(x,0) = \psi_0(x)$. It turns out that we can write ψ in the form

$$\psi(x,t) = \int_{-\infty}^{\infty} K(y,x,t)\psi_0(y)\,dy,$$

where K is the unique solution to (1) satisfying the initial condition (2). The function K is also called the fundamental solution or the Green's function. In physics it is also called the propagator (since it describes how the particle propagates from x_0 to x).

When the potential V is given by

$$V(x) = \frac{m\omega^2}{2}x^2,$$

there is an explicit formula for $K(x,t)$ (see [12]). The Feynman path integral gives a way of finding K for general potentials V. We now sketch a heuristic definition of the path integral.

Suppose our particle moves from x_0 at time 0 to x at time $t > 0$ along a path $\gamma(s)$, where $\gamma(0) = x_0$ and $\gamma(t) = x$. Let $\mathscr{P}$ be the set of all such continuous paths. Assume $s \mapsto \gamma(s)$ is a differentiable function and define the action $S[\gamma, t]$ associated with γ to be the integral of the particle's kinetic energy minus its potential energy:

$$S[\gamma, t] = \int_0^t \left\{ \frac{m}{2} \dot{\gamma}(s)^2 - V(\gamma(s)) \right\} ds. \tag{3}$$

By physical considerations (see [7]), the propagator associated with the path γ has to equal

$$\exp\left(\frac{i}{h} S[\gamma, t] \right).$$

Recalling the probabilistic interpretation of the propagator, it is natural to represent $K(x_0, x, t)$ as the "sum" of the propagators associated with all paths γ in $\mathscr{P}$. Since $\mathscr{P}$ is an uncountable set, this sum is actually an integral of sorts:

$$K(x_0, x, t) = \int_{\mathscr{P}} \exp\left(\frac{i}{h} S[\gamma, t] \right) d\gamma. \tag{4}$$

Note that since the integrand has been defined only for *differentiable* paths and the integral has not been defined at all (what is the meaning of "$d\gamma$"?), this integral is only symbolic, so one needs to provide an additional interpretation of (4) suitable for computational purposes. For this, we refer the reader to [12] and [7].

In our computational setting spacetime is discrete and *finite*, so the path integral reduces to a finite sum. Suppose that $C = \{x_1, \ldots, x_N\}$ is a finite causal set. For fixed $x_i, x_j \in C$, in analogy with (4), we define the probability amplitude of a transition from x_i to x_j by

$$K(x_i, x_j) = \sum_{\gamma \in \mathscr{P}_{ij}} \exp\left(\frac{i}{h} S[\gamma] \right), \tag{5}$$

where $\mathscr{P}_{ij}$ is the set of *all possible* paths (not only chains) γ from x_i to x_j and $S[\gamma]$ is the action associated with γ. It remains to decide on a proper definition for $S[\gamma]$. This is done in the next section.

4. The action and the causality matrix

It is not entirely obvious how the action functional (see the previous section) should be defined in the context of causal sets. We used two approaches: the proper time approach and the binary approach.

4.1. **Proper time.** Recall that in relativity, proper time τ is time measured by a single clock between events that occur at the same place as the clock [4]. It depends not only on the events but also on the motion of the clock between the events in contrast with coordinate time t, which refers to events that occur a distance from the clock. More formally, in special relativity proper time is the pseudo-Riemannian arc-length of the trajectory of a particle (or clock) traveling in 3+1 dimensional spacetime: if $\gamma(t)$, $a \leq t \leq b$, is the trajectory of a particle in spacetime $\mathbb{R}^4$ and $v(t)$ is the coordinate speed of the particle at coordinate time t, then

$$\tau = \int_a^b \sqrt{1 - \frac{v(t)^2}{c^2}}\, dt,$$

where c is the speed of light. If $\gamma(t) = (x(t), y(t), z(t))$ and the speed $v(t)$ is constant, then one obtains

$$c^2 \Delta\tau^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2,$$

where Δu denotes the change in the variable u over $\Delta t = b - a$.

Let us now focus on a 1+1D spacetime represented by a causal set $(C, \prec)$. Take any path in C of length one, that is, let γ be a "jump" from some $x \in C$ to some $y \in C$. Note that we do not necessarily assume that $x \prec y$. (This is in agreement with Feynman's dictum that "a photon does what it wants".) We define the action associated with γ by

$$S[\gamma] = S[x, y] = \frac{2\pi E}{c} \tau(x, y),$$

where $\tau(x, y)$ is the change in proper time corresponding to the transition from x to y. This gives us the probability amplitude of this transition:

$$p(x, y) = \exp\left\{\frac{i}{h} S[x, y]\right\} = \exp\left\{\frac{iE\tau(x, y)}{\hbar c}\right\},$$

where $\hbar = h/2\pi$ is the Dirac constant.

4.2. **Binary approach.** In this approach, we simply set (in the same notation as above)

$$S[x, y] = \frac{2\pi E}{c} \delta(x, y),$$

where

$$\delta(x, y) = \begin{cases} 1, & \text{if} \quad x \prec y \\ 0, & \text{if} \quad x \not\prec y. \end{cases}$$

Then the probability amplitude associated with a path of length one from x to y equals

$$p(x, y) = \exp\left\{\frac{iE\delta(x, y)}{\hbar c}\right\}.$$

This is similar to the so called Feynman checkerboard [5].

4.3. **Causality matrix.** Now assume that C is finite: $C = \{x_1, \dots, x_N\}$. Define an $N \times N$ matrix $A = [a_{ij}]$ by

$$a_{ij} = p(x_i, x_j).$$

This means that the (i, j)-entry of the matrix A is the probability amplitude associated with the one-legged path from x_i to x_j. We call A the causality matrix.

Note that, as usual, the actual probability of an event is the square of the modulus of the corresponding probability amplitude.

It is now natural to ask: what is the probability amplitude associated with a path of length $n > 1$? This answer is given by the following result:

Theorem (Causality matrix)**.** *Let $x_i, x_j \in C$ and $n \geq 1$ be arbitrary. Then the sum of the probability amplitudes associated with paths of length n starting at x_i and terminating at x_j equals the (i, j)-entry of A^n.*

Proof. We will use mathematical induction. For $n = 1$, the statement is true by construction of A.

Assume the statement holds for some $n \geq 1$. Let us show that it holds for $n + 1$. We have that the (i, j)-entry of the matrix A^{n+1} equals:

$$\begin{aligned} A^{n+1}_{i,j} &= (A^n A)_{i,j} \\ &= \sum_{k=1}^{N} A^n_{i,k} A_{k,j}. \end{aligned}$$

By the induction hypothesis, for all $1 \leq i, k \leq N$, $A^n_{i,k}$ is the sum of the probability amplitudes associated with all paths from x_i to x_k of length n. Furthermore, observe that if α, β are two paths such that the terminal point of α coincides with the starting point of β, then

$$p(\alpha * \beta) = p(\alpha)p(\beta),$$

where $\alpha * \beta$ is the concatenation of α and β and $p(\cdot)$ denotes the probability amplitude of a path. It follows that $A^n_{i,k} A_{k,j}$ equals the sum of the probability amplitudes of all paths γ from x_i to x_j of length $n + 1$ of the form $x_i \cdots x_k x_j$, that is, all such γ that visit x_k right before jumping to x_j. Since every path of length $n + 1$ can be broken into the sum of a path of length n and a path of length 1, it follows that

$$\sum_{k=1}^{N} A^n_{i,k} A_{k,j}$$

is the sum of the probability amplitudes of all paths of length $n+1$ from x_i to x_j, as claimed. $\square$

Corollary. *Let $x_i, x_j \in C$ be arbitrary. The probability amplitude of a transition from x_i to x_j of length $\leq n$ equals the (i,j)-entry of the matrix*

$$A^{(n)} = A + A^2 + \cdots + A^n = A(I - A^n)(I - A)^{-1}. \tag{6}$$

4.4. **Computational considerations.** Note that $A^{(n)}$ may not converge as $n \to \infty$, since the norm of the matrix A may not be less than one. We will see later in our results that the normalized probabilities associated with this sum do converge as n increases.

To get any results from our model, we need to decide on an emitter and a detector. The emitter is the point from which the photon is emitted. We usually choose this to be a point in the middle of the causal set. The detector represents a device used to detect the photon. The detector is placed at some specified distance from the emitter. It is expected that the detector's size will be much bigger than the distance between points. Therefore the detector actually covers a distance range. The detector is also there for a given amount of time. Any point in the causal set that is within both the distance range and time range is considered a detector point. See Figure 2.

Each detector point has a space and time coordinate and therefore a specific velocity is required for the photon to reach a particular point. It is not expected that the detector is precise enough to know exactly which detector point absorbed the photon. Therefore, we divide the detector up into several detector regions. The average velocity associated with the points in each region determines the velocity that the detector detects.

From (6), we can find the probability amplitude for the photon to travel to each point in a detector region. We add the probability amplitudes for each point in the region to get the path sum to that detector region. Next, we convert the probability amplitude to a probability. We repeat this process for each detector region. Some normalization of probabilities is usually required at this point to make the probabilities for all detector regions add up to one. Finally, we repeat this process for various photon energies to get our results.

5. Results

5.1. **From the discrete to the continuous.** We first discuss our results in recovering continuous data solely from the combinatorial properties of the causal set.

Proposition 1. *Suppose that a particle moves from a point p to a point q in a causal set $(C, \prec)$ which is faithfully embeddable into a 1+1D spacetime M. Let τ denote the change in proper time corresponding to the transition from p to q. Then*

$$\tau = \sqrt{|[p,q]|}.$$

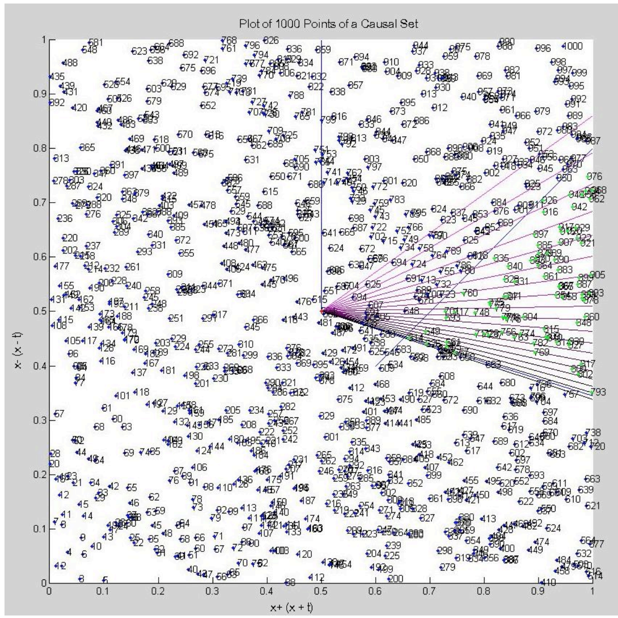

Figure 2. A causal set with 1000 points. The detector is the triangular region partitioned by rays emanating from the center.

Proof. Denote by t and x the change in the coordinate time and the x-coordinate corresponding to the transition from p to q. Then

$$\tau^2 = t^2 - x^2 = \frac{(x_+ + x_-)^2}{4} - \frac{(x_+ - x_-)^2}{4} = x_+ x_-,$$

which is the area of the rectangle R with diagonally opposite vertices p, q in the light-cone coordinates in M, i.e., the interval $[p, q]$. Since $(C, \prec)$ is faithfully embedded into M, the area of R is proportional to the number of points in $[p, q]$ (we can always normalize to make the factor of proportionality equal to one). $\square$

This simple formula produces very good computational results. See Figure 3.

Now assume a *finite* causal set $(C, \prec)$ is faithfully embedded into a *bounded* 1+1D spacetime M, which we assume equals the unit square $[0, 1] \times [0, 1]$ in the light-cone coordinates. Denote by B the point in C corresponding to the origin of the universe and by E the point corresponding to the end of the universe. Then B corresponds to $(0, 0) \in M$ and E to $(1, 1) \in M$.

Proposition 2. *Let $p \in C$ correspond to a point in M with light-cone coordinates (x_+, x_-). The coordinate time t corresponding to the transition*

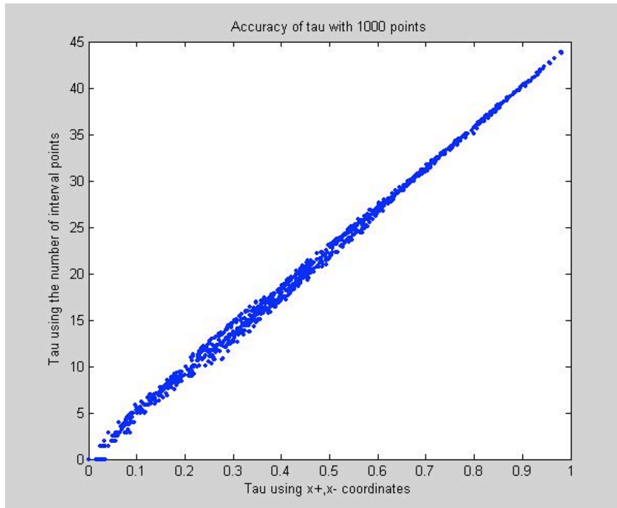

FIGURE 3. Accuracy of the τ-calculation.

from the origin of spacetime to p can be approximated by

$$\frac{1 + N_- - N_+}{2},$$

where $N_- = |[B, p]|$ and $N_+ = |[p, E]|$.

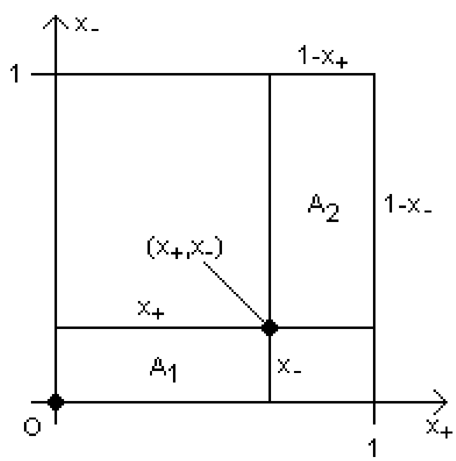

FIGURE 4. Calculating coordinate time.

Proof. Consider Figure 4. Let A_1 denote the area of the rectangle corresponding to the interval between $(0, 0)$ and (x_+, x_-) and A_2 the area of the interval from (x_+, x_-) to $(1, 1)$. Clearly, $A_1 = x_+x_-$ and

$$A_2 = (1 - x_+)(1 - x_-).$$

It follows that

$$\begin{aligned} 1 + A_1 - A_2 &= x_+ x_- - (1 - x_+)(1 - x_-) \\ &= x_+ + x_- \\ &= 2t, \end{aligned}$$

where t is the change in coordinate time from B to p. Since $A_1 \approx N_-$ and $A_2 \approx N_+$, the result follows. □

Figure 5 illustrates that the accuracy of the t-calculation is quite high.

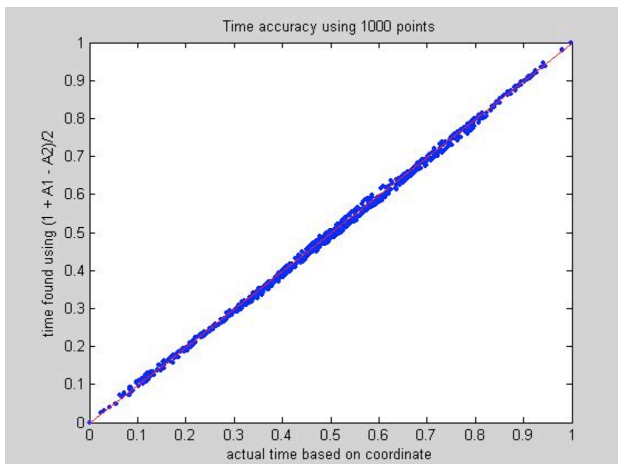

FIGURE 5. Accuracy of the coordinate time calculation.

We can now easily calculate the distance d and velocity v corresponding to a transition from B to p. Namely, since $x^2 = t^2 + \tau^2$, we obtain

$$d = \sqrt{x^2} = \sqrt{t^2 + \tau^2} = \sqrt{\left(\frac{1 + N_- - N_+}{2}\right)^2 + |[B,p]|}.$$

Thus

$$v = \frac{d}{t} = \frac{\sqrt{\left(\frac{1+N_- - N_+}{2}\right)^2 + |[B,p]|}}{\frac{1+N_- - N_+}{2}} == \sqrt{1 + \frac{4\,|[B,p]|}{(1 + |[B,p]| - |[p,E]|)^2}}.$$

Figure 6 shows that the accuracy of the distance calculation is not as high as that of the τ- and t-calculation.

5.2. **Simulation results.** For all our simulations we used MATLAB. Using a method for calculating probability amplitudes based on the causality matrix greatly enhanced our ability to model particles. We are able to use much larger causal sets, a wider range of detector points and expand our

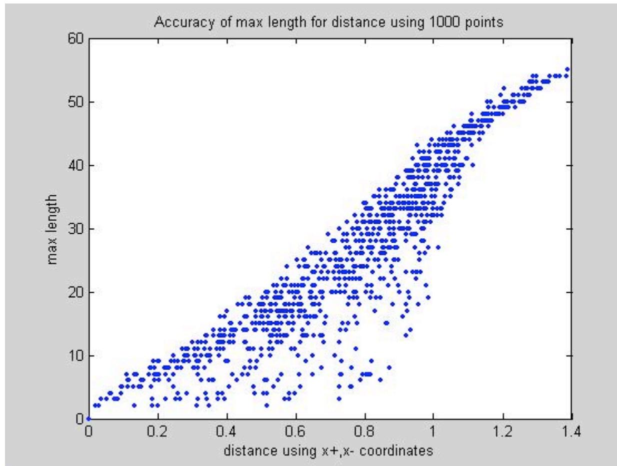

FIGURE 6. Accuracy of the distance calculation.

model to include probabilities for jumps anywhere in space-time, even the non-causally related.

The initial model was based on the adjacency list representation of a graph. To sum the probability amplitudes for all paths between two points, each path had to be enumerated, a process which is exponential in runtime. This made calculating the probability amplitude between just two points computationally infeasible. Introduction of the causality matrix, based off of the adjacency matrix representation of a graph, led to an algorithm that generated the sum of all paths between any two points in the causal matrix in polynomial time.

The bottleneck of the current algorithm is taking the partial sum (6). This is dependent on both the matrix size and the maximum number of jumps. The matrix size is determined directly by the number of points in the causal set, it being a square matrix representing the probability amplitudes of single jumps. The maximum number of jumps is set by the user and affects the algorithm by determining the number of times the matrices must be multiplied.

Holding the maximum number of jumps constant, we determined experimentally that the order of the algorithm is n^3 with causal set size.

As one would expect, holding the causal set size constant, the order of the algorithm is $\log n$ in the maximum jump number.

During our explorations of varying the max jumps, we found that the shape of the curve seemed to be consistent if the maximum number of jumps was above 20. As such, we chose 32 to be our standard maximum jump number.

The significant speed increase of the causality matrix based algorithm allowed for rapid prototyping of additional features. We were able to see near instant results for a 1000 point causal set on a standard desktop while working with our formulas for action.

We now describe the basic steps needed to finish our simulation run. We will focus on a single causal set, a single energy, a single detector distance and a single method for computing action and probability amplitudes. From there, running the model of multiple causal sets, for multiple energies, multiple detector distances and multiple calculation methods can be done by iterating through the changes programmatically or by altering program constants by hand.

For any given point size, we generate the continuum coordinates (x_+, x_-) using the MATLAB native random function that returns values in $(0, 1)$. We force an emitter point in the center by replacing the first point with $(0.5, 0.5)$. Depending on the method for calculating τ, we may also force the points $(0, 0)$ and $(1, 1)$ in a similar fashion.

We then convert the point coordinates to (t, x) and sort by the time coordinate. This step was done as part of a legacy conversion; a similar method in (x_+, x_-) should work. We assign the points numbered labels based on their ascending order. The numbers corresponding to the order represent the points in the causal set. We compare each point to all points following it chronologically to create an adjacency matrix for the points. Recall that a value 1 in the position (i, j) means the j^{th} point is in the causal future of the i^{th} point.

After the adjacency matrix has been generated, we determine a subset of points that are in the detector region with the following algorithm. We check each point against two conditions: (1) The point is within a certain distance from the emitter. This is done with a simple comparison of the x coordinate. (2) The point represents a velocity in the detector range. This is done by comparing the velocity implied by the x and t coordinates of the point ($v = \frac{x}{t}$) and the predetermined velocity range set in the program. Detector points are stored separately, as their adjacency matrix row number, in an array.

The adjacency matrix represents the causal set and contains no continuum information for each point. The detector array is also void of continuum information. This allows various methods for calculating the components of action and probability amplitude without relying on the continuum coordinates. The coordinates are available for calculation methods that require them and for graphical display of the causal set.

The next step in the process is to find the probabilities that each point will be found in the detector strip. To do this, we transformed the adjacency matrix into a causality matrix as discussed earlier.

In its base form, the adjacency matrix is only nonzero where there is a possible jump into the causal future. For many of our simulation runs, we took into account jumps into the causal past as well. To represent this possibility, we added the transpose of the matrix to itself to achieve a symmetric matrix. This matrix was then scaled accordingly to represent the probability amplitude between two points, i and j, rather than just the possibility of jumping. We then put an alternative formula for probability amplitude in the place of any non-diagonal matrix entries that were zero to account for possible jumps between points with no causal relationship. The diagonal was left as zero because it represents a point in space-time transitioning to itself. The matrix rows for each point in the detector array were also set to zero because we consider any point reaching this region already detected and disregard future jumps.

Using the predetermined maximum number of jumps in the program, we then apply formula (6) to our single jump causality matrix. The resulting matrix represents the probability amplitude of the particle traveling between any two points within the given number of jumps. Note that this obtains the probability amplitudes for the particle to be anywhere in our causal set, not just between the emitter and the detector region.

In the case where we do not allow jumps into the causal past, it is possible to calculate the sum of probability amplitudes for all possible path lengths. In this case, we may just continue to add the series, term by term, until a zero term is reached.

However, we only consider the probability amplitudes that we can detect, namely those of the points in our detector array. Each point in the detector array has a probability of detecting the particle equivalent to the square of the probability amplitude. We may then aggregate those probabilities by summing those of detector points that are close to each other in the velocity range. We have done this in two ways. In one, we set a velocity range and sum all detector points within it. In the other, we order the detector points by velocity, and then sum the probabilities of a specific number of the detector points and assign that to the average velocity of each of those detector points. In both cases, we re-normalize the probabilities across all detector points to sum to one. We then have a table of detector velocities and their matching detection probabilities.

As a side note, we now discuss programmatic methods for determining proper time τ, which is the driving force in differentiating the probability amplitudes between causal points.

The method we spent the most time examining was based on the idea that $\tau(i,j)$ could be represented by the number of points in the interval between m and n. To find this, we simply squared the adjacency matrix. Recall that taking the matrix to a power n results in the number of paths of length n between i and j. Each point in the interval results in one path of length two between i and j. Therefore, the square of the adjacency matrix resulted in a matrix representing our τ.

Because τ is relative to the reference frame of the traveling particle, we equated $\tau(i,j)$ to $\tau(j,i)$ such that the number of points in the interval is the same and the sign from the frame of the particle is also the same.

Another method for calculating τ involved examining the maximal chain between the points i and j. To determine this value, we repeatedly multiplied the adjacency matrix with itself, incrementing an accumulator at each place in the matrix where a new longer path was found. We repeated this by the number of points. While this is simple to write in MATLAB, it is time consuming using this particular algorithm. Furthermore, we found this method to be no more accurate (cf., Figure 7) to the τ calculated from the continuum than just using the number of interval points, so we did not further pursue that method.

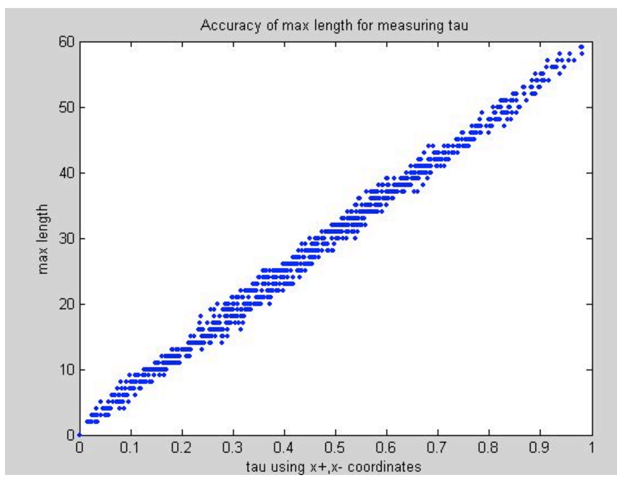

Figure 7. Proper time calculated using maximal chains.

We also performed runs using τ as computed by the continuum coordinates.

Our final method of calculating τ involved setting tau to 1 wherever a causal relationship existed and setting it to zero when two points were not causally related.

Some resulting curves representing the probability distribution of velocities for various energies are shown in Figure 8. In the left frame, we performed $r = 50$ runs with $N = 2500$ points, $e = 2$ energies and paths of maximal length $n = 32$. In the right frame, $N = 1000, r = 75, e = 4$, and $n = 32$. It is hard to draw any definite conclusions from these plots; it is clear that more extensive simulations are needed.

Occasionally, we were fortunate to obtain relatively good-looking distributions as the one in Figure 9.

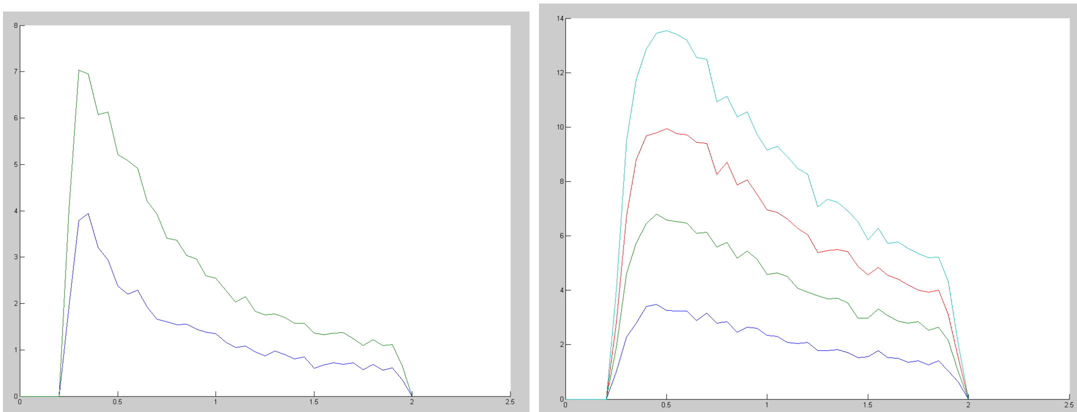

Figure 8. Two typical probability distributions of velocities.

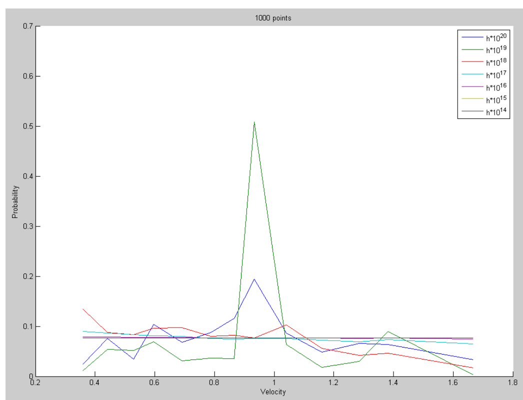

Figure 9. Probability distributions of velocities for various energy levels.

6. Ideas for future work

Our project was but an initial step toward the goal stated in the Introduction. Some suggestions for future work in this direction are:

(a) Set up a better model using more sophisticated physics. In particular, rethink and modify the formula for the action $S[\gamma]$ associated with a path γ.
(b) Find a way to extend the simulation to include *all* possible paths. This involves finding a way around the fact that the causality matrix may have norm greater than one (see Section 4).
(c) Simulate over a larger number of causal sets using more powerful computers.
(d) Extend the simulation and mathematical results to 2+1 and 3+1 dimensional spacetimes.

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