THE PROPAGATION OF LIGHT IN CAUSAL SET THEORY

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Abstract. This is a report based on a student research project conducted at San José State University in the Spring of 2009, which was a continuation of a similar project conducted in the Spring 2008. The goal of both projects was to investigate the possibility that the speed of a photon depends on its energy and thus that the speed of light is not an absolute constant. Both projects were motivated and inspired by the launch of the FERMI Gamma Ray Space Telescope in June 2008, which detects high energy photons coming from Gamma ray bursts. The investigation was done in the context of causal sets implemented in MATLAB. The probability amplitudes were computed using a variant of the Feynman path integral approach. Our work indicates that there is a relationship between a photon's energy and the standard deviation of its velocity.

1. Introduction

This paper is an outcome of the student research project with the same name run by the Center for Applied Mathematics, Computation, and Statistics (CAMCOS) at San José State University in the Spring of 2009. The research project this year was a continuation of the project started in Spring 2008 (see [2]). Both projects were sponsored by Dr. Jeffrey Scargle from the NASA-Ames Research Center. The goal of the project was to explore the possibility that the speed of a photon is a function of its energy. This was done in the context of Causal Set Theory, using a variation on the Feynman Path Integral approach, namely Feynman Path Sums, to compute probability amplitudes. The question was motivated by the NASA launch of the Fermi Gamma Ray Space Telescope on June 11, 2008, which is detecting high-energy photons and may be able to observe possible deviations of the speed of light from its usual value.

Causal Set Theory is one of the competing theories of quantum gravity whose basic postulate is that spacetime is fundamentally discrete. More

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precisely, spacetime is postulated to be a discrete partially ordered set satisfying a local finiteness condition (see Section 2 for details). With this in mind, the goal of the project can be stated as follows. Consider a starting point x and an ending point y in a causal set corresponding to a photon emitted at x and absorbed at y. In classical physics, a transition between x and y can be described by a single path, γ. One can assign to γ a quantity called the action, $S[\gamma]$, which is the integral of the kinetic minus the potential energy. In quantum mechanics, the Feynman path integral (see Section 3) calculates probability amplitudes of a transition between x and y by considering any path γ that the photon could possibly travel between x and y and taking the integral of $\exp(iS[\gamma]/h)$ along all possible γ, where $S[\gamma]$ is the action associated with γ (see Section 4). Of course, there are many problems with this approach: for instance, it is not clear what the meaning of this integral is, since it is done over a very large space of *all* paths. However when dealing with causal sets, all paths are finite, and so we have a better defined sum instead of an integral. We can state the goal of our experiment in the following way:

Goal. *Assign to each possible path a photon can take a physically meaningful value of the action $S[\gamma]$ and use the Feynman integral (i.e., sum) approach to compute the probabilities that the speed of the photon is a function of its energy E.*

The paper is organized as follows. In Section 2, we review the basics of causal set theory and describe the context for our work. Section 3 contains a brief outline of the Feynman path integral approach to quantum mechanics and how we modified it for use in causal sets. In Section 4 we define the action and introduce the concept of a causality matrix. Section 5 will go over our computational methods and the results are discussed in Section 6. Ideas and directions for future research are in Section 7.

Acknowledgments. We would like to thank Dr. Jeffrey Scargle for proposing this topic, his time, patience, and constant help and encouragement. Many thanks also to the director of CAMCOS, Professor Tim Hsu, for keeping us focused and helping with the final presentation.

2. Causal Set Theory

The causal set theory (or program) is one of several approaches to quantum gravity. Quantum gravity is the field of theoretical physics attempting to unify quantum mechanics (the theory that describes three fundamental forces – electromagnetism, weak interaction, and strong interaction), with general relativity, the theory of the fourth fundamental force, gravity. The ultimate goal of quantum gravity is to find a "theory of everything" (TOE). A discussion of quantum gravity is beyond the scope of this paper; for more information, the reader is referred to [11]. The founder and main proponent of causal set theory is Rafael Sorkin.

The basic premise of causal set theory is that spacetime is fundamentally *discrete*. This premise is based on a result of David Malament [10], which states that if f is a map between two past and future distinguishing spacetimes which preserves their causal structure, then the map is a conformal isomorphism (that is, it is a smooth bijection that preserves angles, though not necessarily distances).

Formally speaking, a causal set (or causet) is a set C equipped with a relation $\prec$ with the following properties:

Anti-reflexivity: $x \not\prec x$, for all $x \in C$;
Transitivity: if $x \prec y$ and $y \prec z$, then $x \prec z$;
Local finiteness: for all $x, z \in C$, the set

$$\{y \in C : x \prec y \quad \text{and} \quad y \prec z\}$$

is finite.

That is, a causal set is just a partially ordered set satisfying the local finiteness condition.

A convenient way of representing a causal set $(C, \prec)$ is by means of a directed graph, visualized via a Hasse diagrams. In a Hasse diagram each dot represents an element of a causal set and only the relations not implied by transitivity are drawn as edges (i.e., if $x \prec y$ and $y \prec z$, then one does not draw an edge from x to z). See Figure 1.

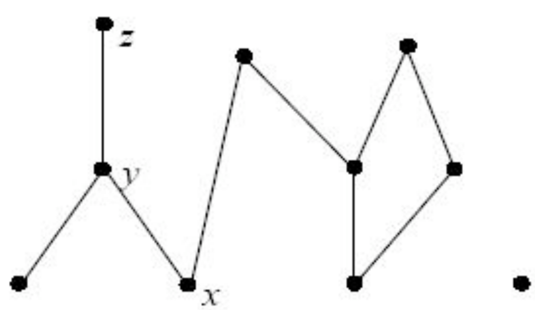

FIGURE 1. The Hasse diagram of a causal set.

Recall that to every graph G with vertices $V = \{v_1, \ldots, v_n\}$ and edges E one can associate a $n \times n$ matrix $A_G = [a_{ij}]$ called the adjacency matrix of G, where $a_{ij} = 1$ if there is an edge from v_i to v_j, otherwise $a_{ij} = 0$. We used this idea to define the concept of a amplitude matrix (see 4; in [2] this matrix was called the causality matrix).

It is natural to ask how such a relatively simple structure as a causal set can be related to a Lorentzian manifold[1] used to represent spacetime in special relativity, i.e., how a causal set C can be *embedded* into a Lorentzian manifold L. An embedding is assumed to be a map $f : C \to L$ such that if $x \prec y$, then $f(x)$ is in the causal past of $f(y)$. The embedding f is called faithful if for every subset $S \subset L$, the number of elements in the preimage $f^{-1}(S)$ is proportional to the volume of S. If such a faithful embedding exists, C is called manifold-like. The *hauptvermutung* (or fundamental conjecture) of causal set theory is that a causal set cannot be faithfully embedded into two spacetimes which are not similar on large scales (a term hard to define precisely).

We remark that not every causal set can be embedded into a Lorentzian manifold; see [7] for an example and a more detailed discussion of this question.

It has been shown that given a causal set $(C, \prec)$ and volume information, it is possible to calculate the dimension, topology, differentiable structure, and metric of the corresponding Lorentzian manifold L (see [1, 6]).

A link in a causal set $(C, \prec)$ is a pair of elements (x, y) such that $x \prec y$ but there is no $z \in C$ such that $x \prec z$ and $z \prec y$. A chain is a sequence $(x_0, x_1, \ldots, x_n)$ of elements of C such that $x_i \prec x_{i+1}$, for all $0 \leq i \leq n-1$. The length of a chain $(x_0, x_1, \ldots, x_n)$ is defined to be n (the number of relations used).

A broader discussion of causal set theory is beyond the scope of this paper (and was beyond the scope of our project). For more details, the reader is referred to [3, 12, 7].

Given a (faithfully embeddable) causal set $(C, \prec)$, we focus on the following questions:

(a) Can continuous physical properties such as coordinate time, proper time, distance, and velocity be recovered using only the combinatorics of a causal set?
(b) How does one adopt the Feynman path integral approach for computing probability amplitudes of various quantum mechanical transitions in C?

Notation and terminology. For background on special relativity, the reader is referred to Feynman's lectures [5].

Given a causal set $(C, \prec)$ and $x, y \in C$ with $x \prec y$, the set

$$[x, y] = \{z \in C : x \prec z \prec y\}$$

is called the interval between x and y.

We will denote the cardinality of a set S by $|S|$.

[1] A Lorentzian manifold is a smooth manifold equipped with a pseudo-Riemannian metric with signature $(1, n-1)$, where n is the dimension of the manifold.

Recall that if $f : C \to L$ is a faithful embedding of a causal set $(C, \prec)$ into a Lorentzian manifold L and $x \prec y$, then (see [7] for more details) $|[x, y]|$ approximately equals the volume (in L) of the intersection of the future light cone on $f(x)$ and the past light cone of $f(y)$.

3. The Feynman Path Integral

Our approach to computing the probability amplitudes of paths in a causal set is based on the Feynman path integral. This subject is too vast to be reviewed here, so we only give a brief, informal discussion that suffices for our purposes. We follow [9].

Suppose that a particle of mass m is at position x_0 on the x-axis at time $t = 0$ and that it has potential energy $V(x_0)$. The particle moves to position x at some future time $t > 0$. According to the laws of quantum mechanics, this move is not deterministic but has a certain probability density $p(x, t)$ of occurring. This probability density is given by $p(x, t) = |K(x_0, x, t)|^2$, where K is a complex number called the probability amplitude. As a function it satisfies the Schrödinger equation

$$ih\frac{\partial K}{\partial t} = -\frac{h^2}{2m}\frac{\partial^2 K}{\partial x^2} + V(x)K, \tag{1}$$

where $\hbar$ is Planck's constant h divided by 2π. Since the particle has position x_0 at time zero, K has to satisfy the initial condition

$$K(x_0, x, 0) = \delta(x - x_0), \tag{2}$$

where δ is the "delta function" defined by

$$\delta(y) = \begin{cases} 0, & \text{if} \quad y \neq 0 \\ \infty, & \text{if} \quad y = 0. \end{cases}$$

More precisely put, δ is a generalized function or a distribution satisfying

$$\int_{-\infty}^{\infty} \delta(x - y) f(y)\, dy = f(x),$$

for every continuous function $f : \mathbb{R} \to \mathbb{R}$.

Suppose we are given a function ψ_0 and would like to construct the solution $\psi(x, t)$ to the Schrödinger equation (1) satisfying the initial condition $\psi(x, 0) = \psi_0(x)$. It turns out that we can write ψ in the form

$$\psi(x, t) = \int_{-\infty}^{\infty} K(y, x, t)\psi_0(y)\, dy,$$

where K is the unique solution to (1) satisfying the initial condition (2). The function K is also called the fundamental solution or the Green's function. In physics it is also called the propagator (since it describes how the particle propagates from x_0 to x).

When the potential V is given by

$$V(x) = \frac{m\omega^2}{2}x^2,$$

there is an explicit formula for $K(x,t)$ (see [9]). The Feynman path integral gives a way of finding K for general potentials V. We now sketch a heuristic definition of the path integral.

Suppose our particle moves from x_0 at time 0 to x at time $t > 0$ along a path $\gamma(s)$, where $\gamma(0) = x_0$ and $\gamma(t) = x$. Let $\mathscr{P}$ be the set of all such continuous paths. Assume $s \mapsto \gamma(s)$ is a *differentiable* function and define the action $S[\gamma, t]$ associated with γ to be the integral of the particle's kinetic energy minus its potential energy:

$$S[\gamma, t] = \int_0^t \left\{ \frac{m}{2}\dot{\gamma}(s)^2 - V(\gamma(s)) \right\} ds. \tag{3}$$

By physical considerations (see [4]), the propagator associated with the path γ has to equal

$$\exp\left(\frac{i}{h}S[\gamma, t]\right).$$

Recalling the probabilistic interpretation of the propagator, it is natural to represent $K(x_0, x, t)$ as the "sum" of the propagators associated with all paths γ in $\mathscr{P}$. Since $\mathscr{P}$ is an uncountable set, this sum is actually an integral of sorts:

$$K(x_0, x, t) = \int_{\mathscr{P}} \exp\left(\frac{i}{h}S[\gamma, t]\right) d\gamma. \tag{4}$$

Note that since the integrand has been defined only for *differentiable* paths and the integral has not been defined at all (what is the meaning of "$d\gamma$"?), this integral is only symbolic, so one needs to provide an additional interpretation of (4) suitable for computational purposes. For this, we refer the reader to [9] and [4].

In our computational setting spacetime is discrete and *finite*, so the path integral reduces to a finite sum. Suppose that $C = \{x_1, \ldots, x_N\}$ is a finite causal set. For fixed $x_i, x_j \in C$, in analogy with (4), we define the probability amplitude of a transition from x_i to x_j by

$$K(x_i, x_j) = \sum_{\gamma \in \mathscr{P}_{ij}} \exp\left(\frac{i}{h}S[\gamma]\right), \tag{5}$$

where $\mathscr{P}_{ij}$ is the set of *all possible* paths (not only chains) γ from x_i to x_j and $S[\gamma]$ is the action associated with γ. It remains to decide on a proper definition for $S[\gamma]$. This is done in the next section.

4. The Action and the Amplitude Matrix

It is not entirely obvious how the action should be defined in the context of causal sets. We used three approaches all differing in their definitions of proper time: standard proper time, modified proper time and s-proper time.

4.1. **Proper Time.**

Recall that in relativity, proper time is time measured by a single clock between events that occur at the same place as the clock. It depends not only on the events but also on the motion of the clock between the events in contrast with coordinate time t, which refers to events that occur a distance from the clock.

We focus on 1+1D spacetime represented by a causal set $(C, \prec)$. Take any path in C of length one, that is, let γ be a jump from some $x_p \in C$ to some $x_q \in C$. Note that we do not necessarily assume that $x_p \prec x_q$. (This is in agreement with Feynman's dictum that a photon does "what it wants".) We then define the action associated with γ by

$$S[\gamma] = S[x, y] = \frac{2\pi E}{c} \tau(x_p, x_q),$$

where $\tau(x_p, x_q)$ is the change in proper time corresponding to the transition from x_p to x_q. In this approach we define the proper time as

$$\tau(x_p, x_q) = \sqrt{\Delta t^2 - \Delta x^2},$$

where Δt is the change in coordinate time and Δx is the change in coordinate position. This gives us the probability amplitude of this transition:

$$\exp\left(\frac{i}{h} S[\gamma]\right) = \exp\left(\frac{iE\tau(x_p, x_q)}{\hbar c}\right),$$

where, as before, $\hbar = h/2\pi$ is the Dirac constant.

4.2. **Modified Proper Time.**

In this approach, we compute (in the same notation as above) the modified proper time as

$$\tau_*(x_p, x_q) = \sqrt{(\Delta t - |\Delta x|)\,(|\Delta t| + |\Delta x|)}.$$

We made the change to modified proper time, τ_*, to try to "recognize" when the transition from $x_p \to x_q$ is causal versus non-causal. Using this definition, τ_* is a positive real number if and only if x_q is in the forward light cone of x_p, otherwise it is a purely imaginary number. Note that $\tau = \tau_*$ when the transition from $x_p \to x_q$ is causal.

This gives us the probability amplitude of this transition:

$$\exp\left(\frac{i}{h} S[\gamma]\right) = \exp\left(\frac{iE\tau_*(x_p, x_q)}{\hbar c}\right).$$

Note, the further x_q is from the forward light cone of x_p the more negative the number $((\Delta t - |\Delta x|)\,(|\Delta t| + |\Delta x|))$ and thus,

$$\begin{aligned}\exp\left(\frac{i}{\hbar}S[\gamma]\right) &= \exp\left(\frac{iE\tau_*(x_p,x_q)}{\hbar c}\right)\\ &= \exp\left(-\frac{E\,(-\Delta t + |\Delta x|)\,(|\Delta t| + |\Delta x|)}{\hbar c}\right)\\ &< 1,\end{aligned}$$

as E, $\hbar$, and c are all positive. Using modified proper time the magnitude of the probability amplitude will decrease with each non-causal jump.

4.3. **s-Proper Time.** In this approach, we compute (in the same notation as above) the s-proper time as

$$\tau_s(x_p,x_q) = \sqrt{(\Delta t - s|\Delta x|)\,(|\Delta t| + s|\Delta x|)},$$

where $0 \le s \le 1$. We mainly experimented with $s = 0$ which gives,

$$\tau_0(x_p,x_q) = \sqrt{\Delta t|\Delta t|} = \begin{cases}\Delta t & \text{if } \Delta t \ge 0\\ i\,|\Delta t| & \text{if } \Delta t < 0.\end{cases}$$

This gives us the probability amplitude of this transition using $s = 0$

$$\exp\left(\frac{i}{\hbar}S[\gamma_{ij}]\right) = \begin{cases}\exp\left(\frac{iE\Delta t}{\hbar c}\right) & \text{if } \Delta t \ge 0\\ \exp\left(-\frac{E\Delta t}{\hbar c}\right) & \text{if } \Delta t < 0.\end{cases}$$

Observe that for $\Delta t < 0$, this expression is a real number < 1.

Setting $s = 0$ expands our "light cone" to the entire upper half plane. This may work more in line with Feynman's dictum that a photon does what it wants as the forward light cone is the entire upper half plane. This approach was taken late in our semester and as such was not fully developed.

4.4. **Amplitude Martix.** Now assume that C is finite: $C = \{x_1, ..., x_N\}$. Define an $N \times N$ matrix $A = [a_{pq}]$ by

$$a_{pq} = \exp\left(\frac{i}{\hbar}S[\gamma_{pq}]\right),$$

where γ_{pq} is the one-legged path from x_p to x_q, i.e., a direct jump from x_p to x_q. (Note that this path may not be an actual path in the causal set C.)

We call A the amplitude matrix. The (i,j)-entry of the amplitude matrix A is the probability amplitude associated with the one-legged path from x_p to x_q. It is a standard result from digraph theory that the n^{th} power of the amplitude matrix contains as entries the probability amplitudes for paths of length n (see also [2]).

Using this fact we define the propagator matrix K as

$$K = (I - A)^{-1} - I. \tag{6}$$

Observe that if $\|A\| < 1$, for some matrix norm $\|\cdot\|$, then K equals the sum of the infinite series $A + A^2 + A^3 + \cdots$. Therefore in that case, the probability amplitude of a transition from x_p to x_q using paths of any length is the (i, j) entry of K. However, we used (6) whether the series $\sum A^n$ converges or *not*. (Observe that (6) makes sense even if that series diverges but 1 is not an eigenvalue of A.) For more discussion of this, see Section 5.

Recall that, as usual, the actual probability of an event is the square of the modulus of the corresponding probability amplitude.

4.5. **Scaling the Amplitude Matrix.** Note that the sum $A + A^2 + \ldots$ may not converge as $n \to \infty$ since the norm of A may not be less than 1. To combat this we decided to attempt to scale the amplitude matrix to ensure convergence.

For each $r, s \in \{1, \ldots, N\}$, $|K_{rs}|^2$ is supposed to represent the probability of the transition from x_r to x_s computed using both causal and non-causal paths. It then makes sense to require that the total probability of a transition from a given x_r to *some* x_s be one, since something is bound to happen in the universe after the event represented by x_r. Therefore K should have the following

Markov Property: *For every* $1 \leq r \leq N$,

$$\sum_{s=1}^{N} |K_{rs}|^2 = 1.$$

In other words, the total probability of a transition is one for any one starting place. To achieve this we defined a scalar α dependent on N which we multiplied by our amplitude matrix. This gave a scaled amplitude matrix αA which would converge. Unfortunately finding an equation for α was a imprecise process due to the inherent randomness of the limited universe in our model. A precise calculation for α would require a very large causal set and as such is likely only possible using analytic or linear algebraic properties of A.

5. Computational Analysis

The mathematical model presented earlier lends itself gracefully to a computational model, as dense adjacency matrices are well understood and there is already a wonderful framework for making complex calculations with just a relatively small amount of human effort.

5.1. **Modeling a Universe.** To observe the behavior of photons in spacetime, we needed to create an environment that demonstrated the predicted characteristics of spacetime. Some key concepts we wished to implement were: approximation of the continuum, discreteness, and Feynman path sums. As it turns out, causal set theory allows for a relatively simple implementation of these concepts.

Approximating the continuum: If we treat each element of a causet C as a point of discrete spacetime, we can say that C represents that region of spacetime. Used in this way C approximates the continuum in that region. Furthermore, we can discover information about this region simply through the combinatorics of C.

Discreteness: The elements of C are discrete by nature; any region represented by C will, by definition, be discrete.

Feynman path sums: Recall that we use Feynman path sums to calculate the probability that a photon will move from one point in spacetime to another, which means we must consider every possible path. The partial order in C allows us to observe all of these paths and also to distinguish between those causal and the non-causal. In the below Hasse diagram, the path from x to y to z is causal, while the path from x to w to z is non-causal. Because the causal relation $\prec$ is anti-reflexive, an event in C cannot cause itself, preventing causal loops within our spacetime.

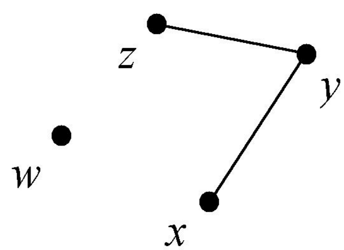

FIGURE 2. Hasse diagram of a causal set.

Our causal sets C were created by randomly sprinkling points into a (very simple) 1+1D Lorentzian manifold, namely, a square. Even though most variables (such as proper time, distance, etc.) can be recovered from the purely combinatorial data given by the causal set (see [2]), due to limited computing power we were forced to use the actual spacetime coordinates in order to speed up our experiments.

To model only causal relations, we used graphs. Recall that to every graph G with vertices $V = \{v_1, \ldots, v_n\}$ and edges E one can associate a $n \times n$ matrix $A_G = [a_{ij}]$ called the *adjacency matrix* of G, where $a_{ij} = 1$ if there is an edge from v_i to v_j, otherwise $a_{ij} = 0$. We used this idea to define the concept of a amplitude matrix (see 4).

An enormous amount of information can be found using the adjacency matrix A_G: While A_G describes all edges of the graph of C, and therefore all causal paths in C of length 1, A_G^2 describes all paths of length 2 (and A_G^n

describes all paths of length n). Using only the A_G matrix we can calculate the number of points in a region of C (and therefore the volume of that region), the number of paths between two points in C, and eventually K, the propagator matrix.

5.2. **Proper Time from Combinatorics.** It was essential to our project that we calculate proper time τ accurately. We have shown how to calculate τ using only spacetime coordinates. While this method produces accurate results, it does not strictly obey our project's goals; ideally, we will calculate τ using only combinatorics of our causality matrix. This is an idea with much room for development, and an area open to further research.

Our method for calculating proper time using combinatorics relies on the assumption that τ_c (this is s-proper time with $s = c$; see 4.3) between two points is proportional to the square root of the number of points in the interval $I = [x, y]$ between two points x and y. In other words, the number of points in I is the number of paths of length 2. In our experiments we scaled the interval by 2. So in our experiments we defined τ_c by

$$\tau_c(x, y) = \sqrt{2L^2(x, y)}$$

where L is the link matrix, that is, the adjacency matrix of the Hasse diagram representing the causal set. (In other words, $L(x, y) = 1$ if there is a path of length one from x to y, otherwise, $L(x, y) = 0$.)

Below are some speed vs. probability histograms produced by this experiment. While this experiment produced a mean speed close to standard c, the speed probability distribution was much different. A further analysis of probability distributions will be covered later in this report.

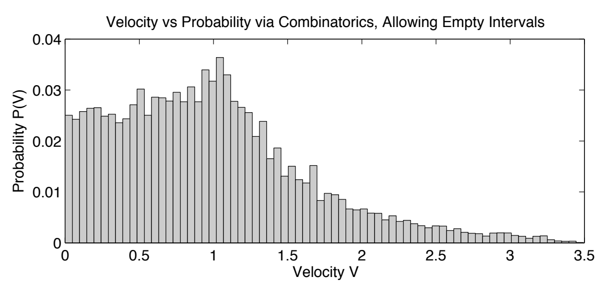

FIGURE 3. Combinatorics experiment run with 800 points; 200 iterations.

Special consideration was made regarding the units we use for our experiments, especially due to the nature of our definition of action S. Values S in the equation $e^{iS/\hbar}$ are desired to be simple enough that our amplitude matrix A will be as well conditioned as possible. Because many of the components of S are based on the Planck scale, it made sense to use Plank units, making

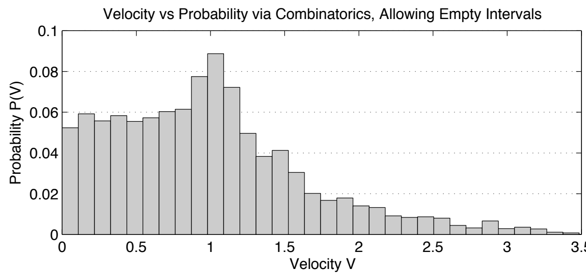

FIGURE 4. Combinatorics experiment run with 800 points; 2400 iterations.

$\hbar$ and the Planck energy each equal to one. The corresponding amplitude matrices are then much better behaved.

5.3. **Computational Methodology.** Since generating our propagator matrix K involved taking a partial sum of a geometric series, we had a few different algorithms to choose from. After running experiments for consistency and observing run-time and memory requirements, we selected one algorithm.

$$K = A + A^2 + ... + A^n \quad (7)$$

$$K = (I - A^n)(I - A)^{-1} - I \quad (8)$$

$$K = (I - A)^{-1} - I \quad (9)$$

The first algorithm was a straightforward partial sum formula (7). The large series of power operations added together gave us an accuracy problem because of floating-point errors in MATLAB, as well as being the worst run-time of the three algorithms we considered.

The second algorithm was the partial sum formula (8). While this was better in both accuracy and run-time, we were still worried about floating-point errors because of that A^n (see 5.4). However, in comparison to our last algorithm, we felt comfortable with the accuracy, but still ended up choosing the last algorithm.

The third algorithm, the one chosen, was the convergence formula for the geometric series (9). While we could not verify that our series actually converged, in practice, this algorithm gave similar results compared to the partial sum formula, so we assume it is correct. This algorithm also gave our fastest run-time, so we ended up choosing it over the partial sum formula.

5.4. **Singular Matrices.** It turned out that the amplitude matrix A is a poorly conditioned matrix for linear algebraic formula such as 7 and 8. Also, we are not guaranteed or even given reason to believe it likely that the geometric series 9 converges. We can see this from figure 5. Note the exponential-like distribution of eigenvalue- and singular values grouped around zero. Our condition numbers typically turn up Inf., 10^{100}, Inf., and NaN, using the one-, two-, Infinity-, and Frobenius-norms, respectively.

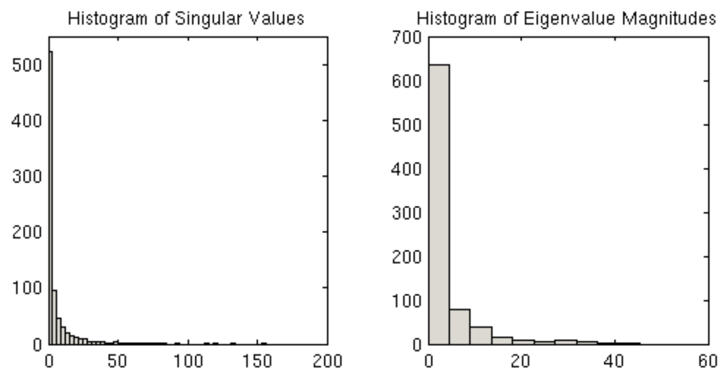

FIGURE 5. Histograms of an amplitude matrix's eigenvalues and singular values.

5.5. **Velocity Probabilities.** To calculate the velocity probabilities, we consider an equal number of points per detector region. We take an average velocity and an absolute square of the total probability of points in a region and concatenate these their respective data vectors. These vectors are what we use to generate our velocity vs. probability plots, which to date are the primary output of our model, used to describe what happened during an aggregated experiment.

Once our model began generating scatter plots of this data with discernible patterns, it became apparent the patterns were convoluted by large amounts of low-probability data. This led us to generate histograms instead.

Since ordinary histograms rely on one-dimensional arrays of data from which they determine relative frequency, we had to generate our own histogram function which takes as its inputs both a set of x-axis values and a set of weighted frequencies (in our case, probabilities). Thus our system adds the probabilities of each pair of velocity-probability data that fall between a histogram bin's width to that bin, then plots all such bins as a bar chart.

5.6. **Interpretation.** In order to analytically describe the distribution we believe we are seeing, a one-sample Kolmogorov-Smirnov- (KS-) test[2]is computed upon the completion of an aggregated experiment and a gamma cumulative distribution function (CDF) and our empirical CDF are plotted against one another.

In order to run a KS-test given a mapping of velocities and their probabilities, the mean $\bar{x}$ and variance s^2 of the data are taken, as described earlier, using the probabilities as weights against the velocities. Then the parameters k and θ are estimated from $\theta' = s^2/\bar{x}$ and $k' = \bar{x}^2/s^2$. Our aggregated experiments typically show a mean and variance of 1.02 and 0.462, corresponding to k and θ values of 2.40 and 0.438.

With the estimated parameters, we can generate our gamma CDF, and by taking the cumulative sum of the probabilities as mapped from velocity, we have our empirical CDF. The MATLAB function `kstest` was modified to allow the use of an empirical CDF in place of a data vector x.

From the experiments corresponding to figure 6, we failed to reject the null hypothesis – that our data falls in a gamma distribution – in 90.0% of trials. However, our P-value is consistently reported to be around 0.229 (with a KS statistic of about 0.0915 and a critical value around 0.117), suggesting the null hypothesis is not necessarily true and our data does not always fall into a gamma distribution.

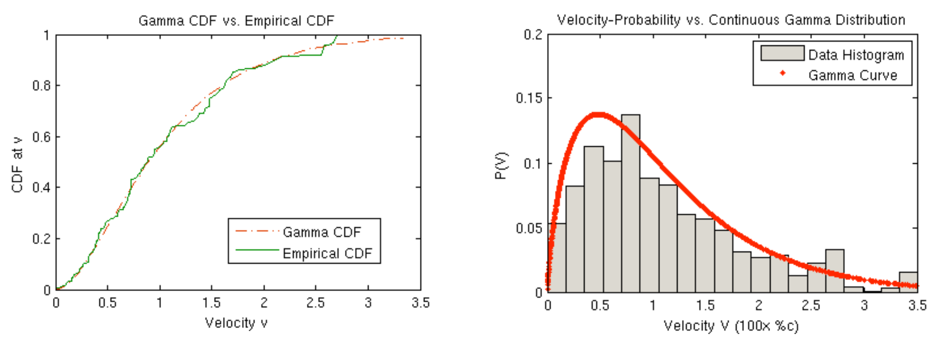

FIGURE 6. (a) Gamma- and empirical- cumulative distribution functions and (b) a gamma distribution superimposed over the empirical data, plotted as a histogram.

5.7. **Optimization and 3+1D.** Since algorithms for multiplying and inverting dense $N \times N$ matrices have an inherent run-time of $O(N^3)$, the number of points used to describe a universe is greatly limited. To best accommodate our run-time for a single experiment while maintaining some assurances of generalization, we limited our experiments to 800 points each and accumulated the output of 20,000 experiments for our distribution analysis.

Beyond this, the greatest improvements to run-time were achieved simply by ensuring good coding practices and by using matlab's built in functionality wherever possible. From the previous model, we see a decrease in

[2] A Kolmogorov-Smirnov test statistic is a minimum distance estimator used to compare distribution functions based on the maximum deviation between two functions. In this case the two functions are an empirical CDF and a gamma CDF. Fig. 6a shows a graphical comparison of two such CDFs from our study.

run-time of a single experiment at 800 points from 38.1 seconds to 4.12 seconds.

Everything discussed in this section thus far pertains to 1+1D space-time. A goal of this project is to move eventually to 3+1D space-time, to better model the way we see our universe.

Moving from 1+1 dimensional space-time to 3+1 requires only a couple changes to our original approach. The first is a departure from light-cone coordinates, which are only defined in 2 dimensions. The second is a 3+1 dimensional definition for the detector, which we have left for future CAMCOS teams.

6. Results

Once our computational model was complete, we ran large aggregated experiments (see fig. 7) to determine if we could show a relationship between the mean velocity and the scaling of a photon's energy. Using very small deviations in scale, we were unable to interpret the results; experiments using scales such as 0.9, 0.99, 0.999, ..., 1.0 resulted in unpredictable behaviour.

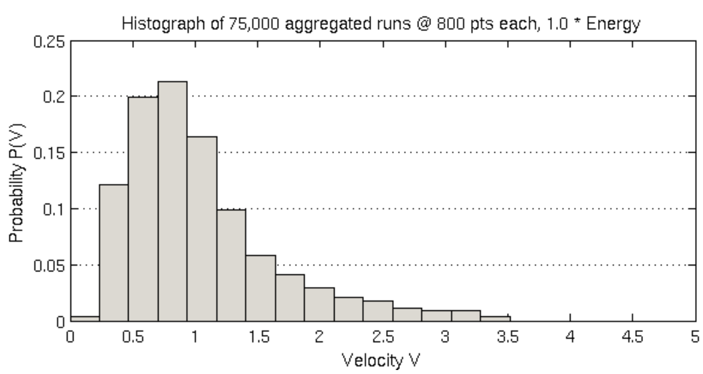

Figure 7. Histogram of probabilities $P(V = v)$ that a photon will travel at velocity v.

However, thanks to a suggestion by Dr. Jeffrey Scargle (NASA-Ames), we completed the same experiment with scales 0.1, 0.2, 0.4, 0.6, and 0.8, which resulted in figure 8 which shows an aparent relationship between a photon's energy and the standard deviation of it's velocity. Also in the figure appears a potential fitting function, given by (10):

$$f(x) = a_0 x^{(k-1)} \frac{e^{-x/\Theta}}{\Gamma(k)\Theta^k}, \tag{10}$$

where $a_0 = 2.7731$, $k = 1.3335$, and $\Theta = 1.3916$. While the fit is good (the correlation coefficient is $R^2 = 0.99214$), there are not enough data points to draw any conclusions about its validity, nor is there yet an explanation as to why this particular function fits as well as it seems to. Further experiments with this model will provide more complete information.

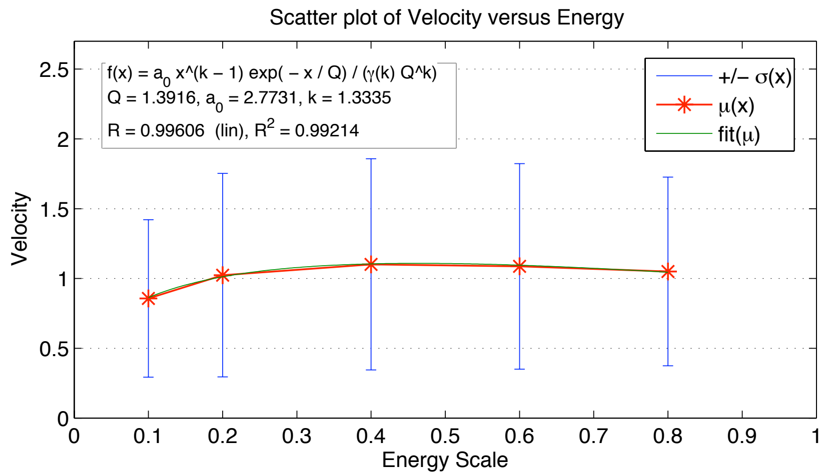

FIGURE 8. Mean from a sample of 20,000 aggregated experiments per data point with error bars at +/− one standard deviation, compared against a scaled gamma probability distribution function. gamma PDF fitted with EzyFit in Matlab®.

6.1. **Comparison with Johnston's work.** In [8], Steven Johnston proposed a quantum mechanical description of particle propagation in a causal set that is very similar to ours. His model involves a discrete path integral, computed using a matrix geometric series, very much like (9). The main difference between his work and ours is that we allow all trajectories, whereas in [8], only those moving forward in time are permitted. Furthermore, Johnston's assigns to each permissible path a *constant* probability amplitude and zero to all the others. Namely, he chooses a constant a as the "amplitude for a particle to 'hop' once along the trajectory from one element to the next" and a constant b as the "amplitude for the particle to 'stop' at one element of the trajectory". This simpler set-up allowed for some more analytical results such as the calculation of the retarded propagator for the Klein-Gordon equation for causal sets sprinkled into 1+1 and 3+1 dimensional spacetime.

By modifying our own model, we implemented Johnson's set-up in MATLAB. A typical histogram of velocity vs. probability is shown in Figure 9.

In most cases, our simulation of Johnston's model did not produce any recognizable distribution. We concluded that the set-up in [8] is not well suited for investigating the dependence of the photon velocity on its energy.

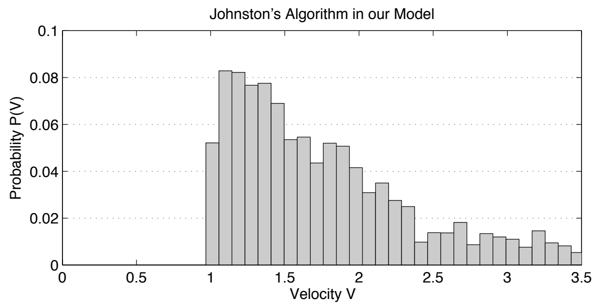

Figure 9. Histogram of speed of light versus probability using Johnston's model

7. Future Research

There are several ideas that we were unable to explore this semester due to computational or time constraints, and would like to suggest as directions for future research.

As the size of our universe or the size of our detector increases, we would expect the variance of our probability distribution to decrease as the speed of light approaches its standard observed value. However, we were unable to verify this asymptotically in changing the size of our universe. This could just be because the number of points difference necessary to observe this is far larger than our model can currently handle. This warrants further research.

In Causal Set Theory all of our values should be determined by the combinatorics of the causal set. However, due to computational difficulties, large runtime increase and ease of access to the continuum data, our current model uses predefined continuum values for time, position, and proper time. Replacing these with combinatorial algorithm generated values would be more true to the theory, but then the computational difficulties would have to be dealt with as well.

We believe that there may be linear algebraic properties of our propagator matrix, K, and our amplitude matrix, A, that could reveal more about the underlying meaning behind the Causal Set Model, and the generation of K. Inspecting these properties may reveal the next step in refining the model.

Since its launch in June 2008, the Fermi Gamma Ray Space Telescope has been collecting data on photons at high energies. One of the goals of the project was to compare this data against our model, and our model is now robust enough that we could get valuable results from that comparison. A priority for future research should be to analyze the model in comparsion to the Fermi data.

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