THE BOUNDARY OF THE GIESEKING TREE IN HYPERBOLIC THREE-SPACE

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Abstract. We give an elementary proof of the Cannon-Thurston Theorem in the case of the Gieseking manifold. We work entirely on the boundary, using ends of trees, and obtain pictures of the regions which are successively filled in by the Peano curve of Cannon and Thurston.

1. Introduction

The Gieseking manifold M is the three-manifold fibered over the circle S^1 with fibre F a punctured torus, and with homological monodromy $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. This monodromy determines the manifold M, which is unorientable, and hyperbolic, with an unoriented cusp.

Let $\mathbf{H}^n$ denote n-dimensional hyperbolic space. Its boundary, $\partial\mathbf{H}^n$, is homeomorphic to the $(n-1)$-sphere S^{n-1}, and its compactification, $\mathbf{H}^n \cup \partial\mathbf{H}^n$, is homeomorphic to the closed n-ball.

The hyperbolic structure of the Gieseking manifold, M, induces a homeomorphism between the universal cover, $\tilde{M}$, and $\mathbf{H}^3$. In particular, we have an action of $\pi_1(M)$ on $\mathbf{H}^3$. Since the fibre F is a punctured torus, it admits several finite volume complete hyperbolic stuctures. Such a hyperbolic structure on F induces a homeomorphism between the universal cover, $\tilde{F}$, and $\mathbf{H}^2$. So every lift of the embedding of F into M induces an embedding of $\mathbf{H}^2$ into $\mathbf{H}^3$, which is $\pi_1(F)$-equivariant.

We give an elementary proof of the Cannon-Thurston Theorem in the particular case of the Gieseking manifold, and hence all finite covers thereof, such as the complement of the figure-eight knot.

Theorem A. [CT] *Every lift to the universal cover of the embedding $F \to M$ induces an embedding $\mathbf{H}^2 \to \mathbf{H}^3$ which extends continously to a $\pi_1(F)$-equivariant quotient map $\Psi\colon\partial\mathbf{H}^2 \to \partial\mathbf{H}^3$.*

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By a *quotient map*, we mean a surjective map between topological spaces, such that the topology of the image space agrees with the resulting quotient topology, so, in particular, the map is continuous.

We remark that, at the time of writing, [CT] leaves the cusped case of the Cannon-Thurston Theorem to the reader, while both [CT] and [Min] give proofs of the cuspless case. In contrast, our arguments are highly specific to the Gieseking example; with more work we could have included also the similar case where the homological monodromy is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, but, beyond that, some our techniques do not seem to apply.

Let G be the group with presentation $\langle a, b, c \mid a^2 = b^2 = c^2 = 1\rangle$, and $\mathcal{T}_G$ the Cayley graph for this presentation, with oriented edges paired off to form unoriented edges. Then $\mathcal{T}_G$ is an unoriented tree whose vertices are trivalent; see Figure 2.2. The set of ends of $\mathcal{T}_G$ is called the *boundary* of $\mathcal{T}_G$, and is denoted $\partial\mathcal{T}_G$. The fundamental group $\pi_1(F)$ of the fibre embeds as a subgroup of index two in G.

We prove Theorem A by considering two G-equivariant maps from $\mathcal{T}_G$, one into $\mathbf{H}^2$, and the other into $\mathbf{H}^3$, which can be extended to maps from $\partial\mathcal{T}_G$ to $\partial\mathbf{H}^2$, and to $\partial\mathbf{H}^3$, respectively.

First we define an action of G on $\mathbf{H}^2 \cup \partial\mathbf{H}^2$. We set

$$\begin{aligned} a(x) &= -1/x, \\ b(x) &= (x-1)/(2x-1), \\ c(x) &= (x-2)/(x-1), \end{aligned}$$

for each x in the completed real line $\overline{\mathsf{R}} = \mathsf{R} \cup \{\infty\}$, with the usual conventions about infinity. This defines an action of G on $\overline{\mathsf{R}}$ by Möbius transformations, and we identify $\partial\mathbf{H}^2$ with $\overline{\mathsf{R}}$. The resulting isometric action of G on $\mathbf{H}^2$ is discrete and faithful. The ideal triangle δ of $\mathbf{H}^2$ having vertices 0, 1, and ∞ in $\partial\mathbf{H}^2$ is a fundamental domain for the action of G. The G-translates of δ determine an ideal tessellation of $\mathbf{H}^2$ called the *Farey tessellation*; see, for example [Ser]. The quotient $G\backslash\mathbf{H}^2$ is a hyperbolic orbifold doubly covered by a punctured torus.

Notice that specifying a point of $\mathbf{H}^2$ is equivalent to specifying a G-equivariant map $\mathcal{T}_G \to \mathbf{H}^2$.

Proposition B. *There exists a G-equivariant quotient map $\tau : \partial\mathcal{T}_G \to \partial\mathbf{H}^2$ which continuously extends any G-equivariant map $\mathcal{T}_G \to \mathbf{H}^2$.*

Next, we define an action of G on $\mathbf{H}^3 \cup \partial\mathbf{H}^3$. We set

$$\begin{aligned} a(z) &= -1/z, \\ b(z) &= (z-1)/((\omega+1)z-1), \\ c(z) &= (z-(\omega+1))/(z-1), \end{aligned}$$

for every z in the Riemann sphere $\overline{\mathsf{C}} = \mathsf{C} \cup \{\infty\}$, where $\omega = \exp(2\pi i/3)$, and we use the standard conventions about infinity. This defines a G-action on $\overline{\mathsf{C}}$ by Möbius transformations, and we identify $\partial\mathbf{H}^3$ with $\overline{\mathsf{C}}$. Again, the resulting isometric action of G on $\mathbf{H}^3$ is discrete and faithful.

Let us describe how this action comes from the hyperbolic structure of the Gieseking manifold. Let γ_1 and γ_2 be free generators of the fundamental group $\pi_1(F)$ of the fibre, F, and let θ be the automorphism of $\pi_1(F)$ such that $\theta(\gamma_1) = \gamma_1^{-1}$ and $\theta(\gamma_2) = \gamma_2^{-1}$. Then G can be viewed as the semidirect product of $\pi_1(F)$ with the group of order two generated by the involution θ. Since θ is in the center of the outer automorphism group of $\pi_1(F)$, it induces an involution of $\pi_1(M)$. So, by Mostow's Rigidity Theorem, G acts isometrically on $\mathbf{H}^3$, and the $\pi_1(F)$-action induced by the G-action is the restriction of the holonomy representation of $\pi_1(M)$. This $\pi_1(F)$-action was studied in detail by Jørgensen and Marden [JM].

Theorem C. *There exists a G-equivariant quotient map $\tilde{\Phi} : \partial\mathcal{T}_G \to \partial\mathbf{H}^3$ which continuously extends any G-equivariant map $\mathcal{T}_G \to \mathbf{H}^3$. Moreover, $\tilde{\Phi}$ factors through the map $\tau : \partial\mathcal{T}_G \to \partial\mathbf{H}^2$ given by Proposition B, to give a G-equivariant quotient map $\Phi : \partial\mathbf{H}^2 \to \partial\mathbf{H}^3$.*

In light of Proposition B, Theorem C follows easily from Theorem A, but we will use the former to prove the latter. The map Φ of Theorem C has the property of the Ψ of Theorem A, because our G-action on $\mathbf{H}^2$ arises from a suitable hyperbolic structure on F.

We recall that a *Jordan curve* in S^2 is a simple closed curve, and a *Jordan region* is a closed subset of S^2 with nonempty interior whose boundary is a Jordan curve. By a *Jordan partition* of S^2 we mean a finite set of Jordan regions, with pairwise disjoint interiors, whose union is all of S^2; similarly, a *Jordan partition* of a Jordan region R of S^2 is a finite set of Jordan regions, with pairwise disjoint interiors, whose union is all of R.

The proof of Theorem C is based on the next result which gives the Jordan partition of $\overline{\mathsf{C}}$ depicted in Figures 1.1a and 1.1b.

FIGURE 1.1A. The Jordan partition of the Riemann sphere $\overline{\mathtt{C}}$. Front.

Theorem D. *There exist three Jordan regions* R_a*,* R_b*, and* R_c *in* $\partial\mathbf{H}^3 = \overline{\mathtt{C}}$*, with pairwise disjoint interiors, satisfying the following.*

(0) $\overline{\mathtt{C}} = R_a \cup R_b \cup R_c$.
(1) $aR_a = R_b \cup R_c$.
(2) $bR_b = R_a \cup R_c$.
(3) $cR_c = R_a \cup R_b$.

The map $\tilde{\Phi}$ of Theorem C will be constructed by using Theorem D to create a *tree of subsets* as follows. We define a map from the set of vertices of $\mathcal{T}_G$ to the set of subsets of $\overline{\mathtt{C}}$. We start with the base vertex 1, which we make correspond to the whole Riemann sphere $\overline{\mathtt{C}}$. Next we make the vertices adjacent to 1, namely a, b, and c, correspond to R_a, R_b, and R_c, respectively. The vertices adjacent to a, other than 1, are ab and ac, which we make correspond to aR_b and aR_c, respectively, and these are contained in R_a, by (1). In a similar way, we make aba correspond to abR_a, which is contained in aR_b, and so on.

Given an end $\mathtt{e} \in \partial\mathcal{T}_G$, we view $\mathtt{e}$ as a *right infinite* reduced word in a, b, c, that is, a reduced word which extends infinitely to the right. We also view $\mathtt{e}$ as

FIGURE 1.1B. The Jordan partition of the Riemann sphere $\overline{\mathsf{C}}$. Back.

the limit of the increasing sequence of its finite initial words, and this sequence corresponds to a decreasing sequence of subsets of $\overline{\mathsf{C}}$. We will show that the intersection of this sequence of subsets is a set which contains a single point, which we then define to be $\tilde{\Phi}(\mathsf{e})$.

From this construction, if $[a] \subset \partial\mathcal{T}_G$ denotes the set of right infinite reduced words starting with a, then $R_a = \tilde{\Phi}[a]$; and similarly with b, or c, in place of a.

We then obtain the following as a consequence.

Corollary E. *Let $\Phi\colon \partial\mathbf{H}^2 \to \partial\mathbf{H}^3$ be the map given by Theorem C. For any $p, q, r, s \in \mathsf{Z}$ such that $ps - qr = \pm 1$, $\Phi([\frac{p}{q}, \frac{r}{s}])$ and $\Phi(\partial\mathbf{H}^2 - [\frac{p}{q}, \frac{r}{s}])$ form a Jordan partition of $\partial\mathbf{H}^3$.*

The paper is organized as follows. In Section 2, we recall some definitions about trees and their boundaries, and we introduce the two examples of trees we shall work with. In Section 3, we introduce the Farey tessellation, which we then use in Section 4 to prove Proposition B. In Section 5, we prove a similar proposition for another tree in a convex subspace of $\mathbf{H}^2$. In Sections 6 and

7, we prove some useful lemmas about hyperbolic three-space and the action of $PGL_2(\mathsf{Z}[\omega])$, where $\omega = \exp(2\pi i/3)$. Theorems C and D, and Corollary E, are proved in Sections 8 and 9. Finally, in Section 11, we prove Theorem A, by using results about the action of $PGL_2(\mathsf{Z}[\omega])$, proved in Section 10.

2. Trees and their boundaries

In this section we recall some definitions, and describe the method of constructing maps from the boundary of a tree by using trees of subsets of a set.

2.1 Definitions. Let $\mathcal{T}$ be a locally-finite tree, that is, $\mathcal{T}$ is a locally-compact one-dimensional simply-connected CW-complex. We can view $\mathcal{T}$ as a combinatorial object or as a metric space, where every edge is isometric to the length one segment $[0,1]$.

The set of vertices of $\mathcal{T}$ is denoted $V(\mathcal{T})$.

From the combinatorial point of view, a *ray* of $\mathcal{T}$ is an infinite sequence of vertices $v_0, v_1, \ldots$ such that v_n is joined to v_{n+1} by an edge, and $v_{n+2} \neq v_n$; we then say that the ray *starts at* v_0. In metric terms, a ray is an isometric map $r\colon[0,+\infty) \to \mathcal{T}$, and it starts at $r(0)$.

Two rays $v_0, v_1, \ldots$ and $w_0, w_1, \ldots$ are said to be *equivalent* if there exist $m_0, n_0 \in \mathsf{Z}$ such that $v_{m_0+n} = w_n$ for each $n \geq n_0$. From the metric point of view, two rays $[0,+\infty) \to \mathcal{T}$ are equivalent if the symmetric difference of their images lies in a compact set.

An *end* of $\mathcal{T}$ is an equivalence class of rays of $\mathcal{T}$ under this equivalence relation.

Let v_0 be a distinguished vertex of T. Every end is represented by a unique ray starting at v_0. For every vertex w of T, let $[w]$ denote the set of ends represented by rays starting at v_0 and passing through w. The *boundary* of $\mathcal{T}$, denoted $\partial\mathcal{T}$, is the topological space whose points are the ends of $\mathcal{T}$, in which $\{[w] \mid w \in V(\mathcal{T})\}$ is a basis for the open topology. This topology does not depend on the choice of the distinguished vertex v_0.

The union $\mathcal{T} \cup \partial\mathcal{T}$ is called the *compactification* of $\mathcal{T}$ and is equipped with a topology such that $\mathcal{T}$ is an open subset, and every end is in the closure of each ray that represents it.

The reader can find further information on ends of trees in Section IV.6 of [DD].

2.2 Definition. Let X be a set.

We write $\mathcal{P}(X)$ to denote the set of all subsets of X.

A *tree of subsets* of X is a tree $\mathcal{T}$ with a distinguished vertex v_0, and a map $\sigma: V(\mathcal{T}) \to \mathcal{P}(X)$ such that, if w lies between v_0 and v, then $\sigma(v) \subseteq \sigma(w)$.

When we described the topology of $\partial\mathcal{T}$, we constructed a tree of subsets $w \mapsto [w]$.

2.3 Definition. Let X be a metric space, and $\mathcal{T}$ a tree with distinguished vertex v_0. A tree of subsets $\sigma: V(\mathcal{T}) \to \mathcal{P}(X)$ is *contracting* if, for every ray $v_0, v_1, v_2, \ldots$, the diameter of $\sigma(v_n)$ tends to zero as n goes to infinity.

2.4 Proposition. *Let X be a compact metric space, $\mathcal{T}$ a tree with distinguished vertex v_0, and $\sigma: V(\mathcal{T}) \to \mathcal{P}(X)$ a tree of closed subsets, that is, $\sigma(v)$ is closed, for every $v \in V(\mathcal{T})$. If σ is contracting, then the map $\phi: \partial\mathcal{T} \to X$, which sends the end represented by $v_0, v_1, v_2, \ldots$ to the unique element of the intersection $\bigcap\limits_{n\geq 0} \sigma(v_n)$, is continuous.*

Proof. Since all the $\sigma(v_n)$ are compact, and the sequence of their diameters tends to zero, $\phi: \partial\mathcal{T} \to X$ is a well-defined map. It remains to check that it is continuous. Let $v_0, v_1, \ldots$ be a ray representing the end $\mathsf{e} \in \partial\mathcal{T}$. For every $\varepsilon > 0$, there is a non-negative integer n_0 such that, for all $n \geq n_0$, $\sigma(v_n)$ is contained in the ball $B_\varepsilon(\phi(\mathsf{e}))$ with center $\phi(\mathsf{e})$ and radius ε; that is what it means for σ to be contracting. In particular, for all $n \geq n_0$, the open subset $[v_n]$ is contained in $\phi^{-1}B_\varepsilon(\phi(\mathsf{e}))$, because $\phi([v_n])$ is contained in $\sigma(v_n)$.

2.5 Notation. We will be working with two different situations where we have a monoid S, and a tree $\mathcal{T}_S$ with vertex set S, and distinguished vertex 1.

2.6 Definitions. We write B to denote a free monoid of rank two, and write $\{f, g\}$ to denote its (unique) free generating set.

Thus the elements of B are the finite words in f and g, including the empty word 1, without any cancellation rule.

Let $\mathcal{T}_B$ be the Cayley graph for B, as in Figure 2.1. Since 1 is the distinguished vertex, every end is represented by a unique right infinite word in f and g.

Assume now that we have a continous action of B on a compact metric space X. This is equivalent to specifying two continuous maps from X to itself, because B is a free monoid. We then define a B-equivariant tree of closed subsets $\sigma: B = V(\mathcal{T}_B) \to \mathcal{P}(X)$, by setting $\sigma(w) = w(X)$ for every $w \in B$. Thus, if σ is contracting, the induced map $\phi: \partial\mathcal{T}_B \to X$ is B-equivariant.

2.7 Definitions. Recall from the Introduction that G denotes the group with presentation $\langle a, b, c \mid a^2 = b^2 = c^2 = 1\rangle$, so G is isomorphic to the free product $C_2 * C_2 * C_2$, where C_2 denotes the order two (cyclic) group.

FIGURE 2.1. The tree $\mathcal{T}_B$ with vertex set B.

The edges of the Cayley graph for this presentation are paired, since for each edge there is another edge joining the same two vertices, but going in the opposite direction. We treat each such pair of oriented edges as a single unoriented edge, and so obtain a tree $\mathcal{T}_G$ with vertex set G, as in Figure 2.2. Each vertex is trivalent.

As 1 is the distinguished vertex, every end is represented by a right infinite reduced word in a, b and c. (Recall that a word in a, b and c is *reduced* if any two consecutive letters are different, that is, all occurrences of a^2, b^2 and c^2 are cancelled.)

Now suppose that G acts continuously on a compact metric space X. Suppose also that we have three closed subsets R_a, R_b, and R_c of X satisfying the following:

(0) $X = R_a \cup R_b \cup R_c$;
(1) $aR_a = R_b \cup R_c$;
(2) $bR_b = R_a \cup R_c$;
(3) $cR_c = R_a \cup R_b$.

We now proceed to define a tree of subsets $\sigma : V(\mathcal{T}_G) = G \to \mathcal{P}(X)$. We set $\sigma(1) = X$, and, for any non-trivial $w \in G$, we set $\sigma(w) = wdR_d$ where $d \in \{a, b, c\}$ is the last letter of w, when written as a reduced word. It follows from (1), (2), and (3) that σ is a tree of subsets.

FIGURE 2.2. The tree $\mathcal{T}_G$ with vertex set $G = \langle a, b, c | a^2 = b^2 = c^2 = 1\rangle$.

When $X = \partial\mathcal{T}_G$, if we take $R_a = [a]$, $R_b = [b]$, and $R_c = [c]$, then (0), (1), (2), and (3) are satisfied:

(0′) $\partial\mathcal{T}_G = [a] \cup [b] \cup [c]$;
(1′) $a[a] = [b] \cup [c]$;
(2′) $b[b] = [a] \cup [c]$;
(3′) $c[c] = [a] \cup [b]$.

Thus we have a natural tree of subsets of $\partial\mathcal{T}_G$.

2.8 Proposition. *Let X be a compact metric space and $\sigma : V(\mathcal{T}_G) = G \to \mathcal{P}(X)$ a tree of subsets as in Definition 2.7. If σ is contracting, then the induced map $\phi : \partial\mathcal{T}_G \to X$ is a G-equivariant quotient map.*

Proof. The map ϕ is surjective, since equality holds in (0), (1), (2) and (3). Moreover, ϕ maps closed subsets of $\partial\mathcal{T}_G$ to closed subsets of X, because $\partial\mathcal{T}_G$ is compact and X is Hausdorff. Thus ϕ is a quotient map.

The map ϕ is G-equivariant because σ has the property that, for all $v, w \in G$, if w is not completely cancelled in vw, then $\sigma(vw) = v\sigma(w)$.

3. The Farey tessellation of $\mathbf{H}^2$

In this section we describe the Farey tessellation, and the action of G on $\mathbf{H}^2$.

3.1 Definitions. Recall that $\mathbf{H}^2$ denotes the hyperbolic plane, and $\partial\mathbf{H}^2$ its boundary.

We identify $\mathbf{H}^2$ with the half-plane $\{z \in \mathsf{C} \mid \mathrm{Im}(z) > 0\}$, and $\partial\mathbf{H}^2$ with the completed real line $\overline{\mathsf{R}} = \mathsf{R} \cup \{\infty\}$.

The isometry group of $\mathbf{H}^2$ is naturally isomorphic to the group of Möbius transformations of $\partial\mathbf{H}^2$, as follows. Each isometry of $\mathbf{H}^2$ extends continuously to $\mathbf{H}^2 \cup \partial\mathbf{H}^2$ in a unique way, and the resulting action on $\partial\mathbf{H}^2$ is a Möbius transformation. Moreover every Möbius transformation of $\partial\mathbf{H}^2$ arises in this way, from a unique isometry. The group of orientation-preserving Möbius transformations of $\overline{\mathsf{R}}$ is isomorphic to $PSL_2(\mathsf{R})$. We represent elements of $PSL_2(\mathsf{R})$ in the form $\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and such an element corresponds to the isometry and Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}$$

for all $z \in \mathbf{H}^2 \cup \partial\mathbf{H}^2$, with the usual conventions about infinity.

For any subset X of $\partial\mathbf{H}^2$, the *convex hull* of X is the smallest convex subset of $\mathbf{H}^2$ containing X in its closure.

An *ideal* triangle in $\mathbf{H}^2$ is a finite volume triangle whose vertices lie in $\partial\mathbf{H}^2$; that is, an ideal triangle is the convex hull of three points of $\partial\mathbf{H}^2$.

The *Farey tessellation* of $\mathbf{H}^2$ is a tessellation by ideal triangles with vertices lying in $\mathbf{Q} \cup \{\infty\}$, as in Figure 3.1. (Here, and throughout, we will usually depict the upper half-plane as a disc, via a conformal transformation.) Two rational numbers $\frac{p}{q}$ and $\frac{r}{s}$ in lowest terms (allowing also $\infty = \frac{\pm 1}{0}$) are joined by an edge in the Farey tessellation if and only if $ps - qr = \pm 1$. Every edge having vertices $\frac{p}{q}$ and $\frac{r}{s}$ is adjacent to the triangles obtained by adding in the vertices $\frac{p+r}{q+s}$ and $\frac{p-r}{q-s}$. Using the Euclidean algorithm, one can prove that this defines a tesselation, and it is clear that the Farey tessellation is preserved by the action of $PSL_2(\mathsf{Z})$.

3.2 Definitions. Consider the action of $G = \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$ on $\mathbf{H}^2$ which we described in the Introduction. This action is defined by mapping generators of G to $PSL_2(\mathsf{Z})$ as follows:

$$a \mapsto \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad b \mapsto \pm\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \qquad \text{and} \qquad c \mapsto \pm\begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.$$

FIGURE 3.1. The Farey tessellation of $\mathbf{H}^2$.

Since the image lies in $PSL_2(\mathsf{Z})$, the G-action is discrete and preserves the Farey tessellation. Notice that a (resp. b, resp. c) acts as an order two rotation around a point in the edge with vertices 0 and ∞ (resp. 0 and 1, resp. 1 and ∞).

Let δ denote the ideal triangle of the Farey tessellation having vertices 0, 1 and ∞. By Poincaré's Theorem, we have the following.

3.3 Lemma. *The G-action on $\mathbf{H}^2$ is faithful, and has δ as a fundamental domain.*

3.4 Definitions. The *dual tree* of the Farey tessellation is the tree whose vertices are the ideal triangles of this tessellation, in which two vertices of the dual tree are joined by an edge if and only if the corresponding triangles have a common edge. It follows, from Lemma 3.3, that the dual tree of the Farey tessellation is isomorphic to our Cayley graph $\mathcal{T}_G$. Given a point x in the interior of δ, the map

$$g \in G \mapsto g(x) \in \mathbf{H}^2$$

FIGURE 3.2. A copy of $\mathcal{T}_G$ in $\mathbf{H}^2$.

induces a G-equivariant embedding $i\colon\mathcal{T}_G \to \mathbf{H}^2$ which realizes the isomorphism between $\mathcal{T}_G$ and the dual tree of the Farey tessellation; see Figure 3.2.

We end this section by recording a result from hyperbolic geometry which we will use to study extensions to boundaries. It is well-known and easy to prove.

3.5 Lemma. *Let $(I_n)_{n\in\mathsf{N}}$ be a decreasing sequence of intervals in $\overline{\mathsf{R}}$, whose lengths tend to zero, and whose intersection is a singleton $\{x_\infty\}$. Let $(x_n)_{n\in\mathsf{N}}$ be a sequence in $\mathbf{H}^2$. If, for each $n\in\mathsf{N}$, x_n is contained in the convex hull of I_n, then the sequence of the x_n tends to x_∞.*

4. THE BREAKUP OF $\overline{\mathsf{R}}$

In this section we prove Proposition B, and use the Farey tessellation to construct the map $\tau\colon\partial\mathcal{T}_G\to\partial\mathbf{H}^2$.

4.1. *Proof of Proposition B.* We shall show that there exists a map $\tau:\partial\mathcal{T}_G\to\partial\mathbf{H}^2$ which is the continuous extension of a G-equivariant map

$i\colon \mathcal{T}_G \to \mathbf{H}^2$. We first construct τ, and then recover i. We define τ by applying Proposition 2.4 to a contracting tree of subsets of $\partial\mathbf{H}^2$. Consider the following subsets of $\overline{\mathsf{R}} = \partial\mathbf{H}^2$:

$$R_a = [-\infty, 0], \qquad R_b = [0,1], \qquad \text{and} \qquad R_c = [1, +\infty].$$

It is easily checked that these subsets satisfy (0), (1), (2), and (3), because their end-points are the vertices of the fundamental domain δ. By Definition 2.7, this choice of R_a, R_b, and R_c defines a tree of subsets $\sigma\colon V(\mathcal{T}_G) \to \mathcal{P}(\partial\mathbf{H}^2)$. Thus $\sigma(1) = \overline{\mathsf{R}}$, $\sigma(a) = R_a$, $\sigma(b) = R_b$, $\sigma(c) = R_c$, $\sigma(ab) = aR_b$, and so on.

4.2 Lemma. *The tree of subsets $\sigma\colon V(\mathcal{T}_G) \to \mathcal{P}(\partial\mathbf{H}^2)$ is contracting.*

Proof. For every vertex $v \in V(\mathcal{T}_G) = G$, the set $\sigma(v)$ is an interval of $\overline{\mathsf{R}}$ whose end-points are the ideal vertices of an edge of the Farey tessellation. That is, $\sigma(v) = [\frac{p}{q}, \frac{r}{s}]$ for certain $p, q, r, s \in \mathsf{Z}$ such that $ps - qr = \pm 1$. This implies that the length of $\sigma(v)$ is either infinite, or the inverse of an integer, because $\frac{r}{s} - \frac{p}{q} = \pm\frac{1}{qs}$. Moreover if the length of $\sigma(v)$ is infinite, $\sigma(v)$ is either $[-\infty, m]$ or $[m, +\infty]$, for some $m \in \mathsf{Z}$.

Let $v_0, v_1, \ldots$ be a ray in $\mathcal{T}_G$ starting at $v_0 = 1$. The sequence of intervals $(\sigma(v_n))_{n\in\mathsf{N}}$ is strictly decreasing, and the lengths are inverses of integers or infinite. Thus, except in the case where ∞ is contained in $\sigma(v_n)$ for each $n > 0$, the sequence of lengths of the $\sigma(v_n)$ tends to zero. When $\infty \in \sigma(v_n)$, either $\sigma(v_n)$ is of the form $[-\infty, m_n]$ and m_n is a decreasing sequence of integers, or it is of the form $[m_n, +\infty]$ and m_n is an increasing sequence of integers. In both cases, for any Riemannian metric of $\overline{\mathsf{R}}$, the sequence of lengths of the $\sigma(v_n)$ tends to zero.

4.3. We return to the proof of Proposition B.

Since σ is contracting, Proposition 2.4 implies that we get a continuous G-equivariant map $\tau : \partial\mathcal{T}_G \to \partial\mathbf{H}^2$, which is a quotient map by Proposition 2.8.

Let $i : \mathcal{T}_G \to \mathbf{H}^2$ be any G-equivariant map. Let $(v_n)_{n\in\mathbf{N}}$ be a sequence of vertices of $\mathcal{T}_G$ with limit an end $\mathsf{e} \in \partial\mathcal{T}_G$. We claim that the sequence of $i(v_n)$ tends to $\tau(\mathsf{e})$. By G-equivariance, $i(v_n) = v_n i(1)$, and we recall that 1 is the distinguished vertex of $\mathcal{T}_G$. We may assume that $i(1) \in \delta$, because the convergence of the sequence $(v_n(x))_{n\in\mathsf{N}}$, and its limit, is independent of the point $x \in \mathbf{H}^2$. Since $i(1) \in \delta$, $i(v_n) \in v_n\delta$, so $i(v_n)$ belongs to the convex hull of $\sigma(v_n)$. Since σ is contracting, and $\bigcap_{n\in\mathsf{N}} \sigma(v_n) = \{\tau(\mathsf{e})\}$, the sequence of $i(v_n)$ tends to $\tau(\mathsf{e})$, by Lemma 3.5. This completes the proof of Proposition B.

4.4 Lemma. *The map* $\tau : \partial\mathcal{T}_G \to \overline{\mathsf{R}}$ *of 4.3 is finite-to-one. The preimage* $\tau^{-1}(\xi)$ *has two points if* $\xi \in \mathsf{Q} \cup \{\infty\}$*, and one point if* $\xi \in \mathsf{R} - \mathsf{Q}$*.*

Proof. Let $\xi \in \mathsf{R} - \mathsf{Q}$ and $\mathsf{e} \in \partial\mathcal{T}_G$ such that $\tau(\mathsf{e}) = \xi$. We claim that e is unique. Let $v_0, v_1, \ldots$ be the unique ray in $\mathcal{T}_G$ starting at $v_0 = 1$, and representing e. The vertex v_1 is determined by whether ξ belongs to $\sigma(a) = R_a = [-\infty, 0]$, $\sigma(b) = R_b = [0, 1]$, or $\sigma(c) = R_c = [1, +\infty]$, and the choice of a, b, or c is forced, because ξ is irrational. Similarly, for each $n \in \mathsf{N}$, there is only one vertex v_n in $\mathcal{T}_G$ at distance n from $v_0 = 1$ such that $\xi \in \sigma(v_n)$, because ξ is irrational. We conclude that v_n is uniquely determined by ξ for every n, and, hence, so is e.

If ξ is rational or infinity, it is clear that, for all $n \in \mathsf{N}$, there are at most two vertices v_n and v'_n at distance n from $v_0 = 1$ such that $\sigma(v_n)$ and $\sigma(v'_n)$ contain ξ; since ξ is a vertex in the Farey tesselation, there do exist two such vertices if n is sufficiently large. In a ray $v_0, v_1, \ldots$, the vertices v_0 and v_n determine the vertices in between them, so there are exactly two ends in the preimage of ξ.

We write $(abc)^\infty$ to denote the right infinite word $abcabc\cdots$, viewed as an end of $\mathcal{T}_G$, and similarly for $(cba)^\infty$.

4.5 Proposition. *Let* G *act continuously on a topological space* X*, and let* $\tilde{h} : \partial\mathcal{T}_G \to X$ *be a continuous* G*-equivariant map. Then* $\tilde{h}$ *factors through the map* $\tau : \partial\mathcal{T}_G \to \overline{\mathsf{R}}$ *of 4.3, to give a continuous map* $h : \overline{\mathsf{R}} \to X$*, if and only if* $\tilde{h}((abc)^\infty) = \tilde{h}((cba)^\infty)$*.*

Proof. Direct calculation shows that $abc(x) = x - 3$ for all $x \in \overline{\mathsf{R}}$. Thus $\infty \in \overline{\mathsf{R}}$ is the only fixed point of abc. Since τ is G-equivariant, and $(abc)^\infty$ is fixed by abc, we see that $\tau((abc)^\infty) = \infty$. But the same holds for $cba = (abc)^{-1}$, so $\tau((abc)^\infty) = \tau((cba)^\infty)$. Hence, if $\tilde{h}$ factors through τ, then $\tilde{h}((abc)^\infty) = \tilde{h}((cba)^\infty)$.

Now suppose that $\tilde{h}((abc)^\infty) = \tilde{h}((cba)^\infty)$.

We claim that for every rational ξ there exists $w \in G$ such that $\xi = w(\infty)$. Since the dual tree of the Farey tessellation is isomorphic to $\mathcal{T}_G$, there exists $w' \in G$ such that ξ is a vertex of the ideal triangle $w'(\delta)$, where δ is the ideal triangle with vertices $0, 1, \infty$. Moreover $a(0) = c(1) = \infty$, so there exists $w \in G$ with $\xi = w(\infty)$.

From the previous paragraph, and the G-equivariance of τ and $\tilde{h}$, we deduce that $\tilde{h}$ factors through a map $h : \overline{\mathsf{R}} \to X$, which is continuous because τ is a quotient map.

5. The breakup of $[0,+\infty]$

In this section, we describe an action of B on $[0,+\infty]$, and we study a B-equivariant map $\nu\colon\partial\mathcal{T}_B \to [0,+\infty]$, using the Farey tessellation.

5.1 Notation. Let $\mathbf{H}^2_+$ denote the convex hull of $[0,\infty]$ in $\mathbf{H}^2$, and consider the restriction of the Farey tessellation to $\mathbf{H}^2_+$, as in Figure 5.1.

Figure 5.1. The restriction of the Farey tessellation to $\mathbf{H}^2_+$.

Consider the action of B on $\mathbf{H}^2_+ \cup [0,\infty]$ in which the free generators act by $f(x) = x+1$, $g(x) = \frac{x}{x+1}$, for all $x \in [0,\infty]$; this corresponds to

$$f \mapsto \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } g \mapsto \pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Since $f([0,\infty]) = [1,\infty]$, and $g([0,\infty]) = [0,1]$, are contained in $[0,\infty]$, and the two $\pm$-matrices belong to $PSL_2(\mathsf{Z})$, the B-action respects the restriction of the Farey tessellation to $\mathbf{H}^2_+$.

5.2 Lemma. *The ideal triangle δ whichs has vertices* 0, 1, *and* ∞ *is a fundamental domain for the B-action on* $\mathbf{H}^2_+$.

Proof. In order to prove that $B\delta = \mathbf{H}^2_+$, we consider the group Γ generated by $\gamma_1 = fg$ and $\gamma_2 = gf$. Let D be the ideal quadrilateral of $\mathbf{H}^2$ with vertices 0, 1, ∞ and -1. Since γ_1 and γ_2 identify the faces of D, Poincare's Theorem yields that D is a fundamental domain for the action of Γ on $\mathbf{H}^2$. In particular, $\mathbf{H}^2 = \Gamma D$.

Recall from the definition of the G-action on $\mathbf{H}^2$ that we are using the conformal transformation $a(x) = -1/x$. From the formulas

$$a^2 = Id, \qquad f^{-1} = aga, \qquad g^{-1} = afa, \qquad faf = g \quad \text{and} \quad gag = f,$$

we conclude that every element of Γ can be written in the form w, or awa, for some $w \in B$. Moreover, $\mathbf{H}^2 = B\delta \cup aB\delta$, since

$$fa\delta = \delta, \qquad ga\delta = \delta \quad \text{and} \quad D = \delta \cup a\delta.$$

Hence, $B\delta = \mathbf{H}^2_+$, because $B\delta \subseteq \mathbf{H}^2_+$ and the regions $\mathbf{H}^2_+$ and $a\mathbf{H}^2_+$ have disjoint interiors.

Since the Farey tessellation is preserved by $PSL_2(\mathsf{Z})$, it remains to show that distinct elements give distinct δ-translates. This follows immediately from the fact that the regions δ, $f\mathbf{H}^2_+$ and $g\mathbf{H}^2_+$ have disjoint interiors.

In particular, the dual graph of the restriction of the Farey tessellation to $\mathbf{H}^2_+$ is isomorphic to the Cayley graph $\mathcal{T}_B$ of B. For a point x in the interior of δ, the map

$$w \in B \mapsto w(x) \in \mathbf{H}^2_+$$

induces a B-equivariant embedding $j : \mathcal{T}_B \to \mathbf{H}^2_+$, which realizes the isomorphism between $\mathcal{T}_B$ and the dual tree of the restriction of the Farey tessellation to $\mathbf{H}^2_+$; see Figure 5.2.

5.3 Proposition. *There exists a B-equivariant quotient map $\nu : \partial\mathcal{T}_B \to [0, \infty]$ such that every B-equivariant map $j : \mathcal{T}_B \to \mathbf{H}^2_+$ extends to ν.*

Moreover the set $\nu^{-1}(\xi)$ has two elements if $\xi \in \mathsf{Q} \cap (0, \infty)$, and one element if $\xi \in [0, \infty] - (\mathsf{Q} \cap (0, \infty))$.

Proof. The proof of this proposition is analogous to the proof of Proposition B, the only difference being in the construction of the tree of subsets. In this case, we use Definition 2.6 to get a B-equivariant tree $\sigma : V(\mathcal{T}_B) \to \mathcal{P}(X)$ defined by $\sigma(v) = v([0, \infty])$, for every $v \in V(\mathcal{T}_B) = B$. The proof that σ is contracting is analogous to that of Lemma 4.2, and the affirmation about the cardinality of the set $\nu^{-1}(\xi)$ is proved in the same way as Lemma 4.4. Note

FIGURE 5.2. An embedding of $\mathcal{T}_B$ in $\mathbf{H}^2_+$.

that, since 0 and ∞ are the end points of $[0,\infty]$, the preimages $\nu^{-1}(0)$ and $\nu^{-1}(\infty)$ have only one element.

Let f^∞ denote the right infinite word $ff\cdots$, which is viewed as an end of $\mathcal{T}_B$.

5.4 Proposition. *Let B act continuously on a topological space X, and let $\tilde{h}\colon\partial\mathcal{T}_B \to X$ be a continuous B-equivariant map. Then $\tilde{h}$ factors through the map ν to give a continuous map $h:[0,+\infty] \to X$ if and only if $\tilde{h}(g^2f^\infty) = \tilde{h}(fgf^\infty)$.*

Proof. The only elements of $\tau^{-1}(1)$ are g^2f^∞ and fgf^∞, because ∞ is the only point fixed by f, and $g^2(\infty) = fg(\infty) = 1$.

By Lemma 5.2, for each $\xi \in \mathtt{Q} \cap (0,\infty)$ there exists $w' \in B$ such that ξ is a vertex of $w'\delta$. Since the vertices of δ are 0, 1, and ∞, we see that ξ is equal to $w'(1)$, $w'(0)$, or $w'(\infty)$. By taking a word w shorter that w' if necessary, we may write $\xi = w(1)$, because $f(\infty) = \infty$, $g(\infty) = 0$, and $f(0) = g(0) = 1$. Thus every rational in $(0,\infty)$ is of the form $w(1)$, for some $w \in B$.

Now we complete the argument as in the proof of Proposition 4.5.

6. The action of $PGL_2(\mathsf{Z}[\omega]) \rtimes C_2$ on $\partial\mathbf{H}^3$

In this section we prove some useful facts about the action of the isometry group of hyperbolic 3-space $\mathbf{H}^3$, particularly for isometries in $PGL_2(\mathsf{Z}[\omega]) \rtimes C_2$, where $\omega = \exp(2\pi i/3)$. Corollary 6.5 gives a criterion for proving that a tree of subsets of $\overline{\mathsf{C}}$ is contracting, which uses the arithmeticity of $PGL_2(\mathsf{Z}[\omega])$.

6.1. We identify $\mathbf{H}^3$ with the half-space

$$\mathbf{H}^3 = \{z + t\,j \mid z \in \mathsf{C}, t \in \mathsf{R}, t > 0\},$$

and identify its boundary, $\partial\mathbf{H}^2$, with the completed complex line $\overline{\mathsf{C}} = \mathsf{C} \cup \{\infty\}$. The compactification $\mathbf{H}^3 \cup \partial\mathbf{H}^3$ is homeomorphic to a closed three-ball. Here $\mathsf{C} + \mathsf{C}\,j$ can be identified with the quaternions, and $\partial\mathbf{H}^3$ is the completion of the complex line corresponding to taking $t = 0$, and we use the usual conventions about infinity.

The isometries of $\mathbf{H}^3$ extend continuously to Möbius transformations of $\partial\mathbf{H}^3$, and this extension induces an isomorphism between the isometry group of $\mathbf{H}^3$ and the group of Möbius transformations of $\partial\mathbf{H}^3$. The orientation-preserving isometry group of $\mathbf{H}^3$ is identified with $PSL_2(\mathsf{C})$. An element $\pm\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of $PSL_2(\mathsf{C})$ corresponds to the isometry and Möbius transformation

$$z + t\,j \in \mathbf{H}^3 \cup \partial\mathbf{H}^3 \mapsto (a(z + t\,j) + b)(c(z + t\,j) + d)^{-1} \in \mathbf{H}^3 \cup \partial\mathbf{H}^3.$$

We remark that $PSL_2(\mathsf{C})$ is isomorphic to $PGL_2(\mathsf{C})$, but the above formula for isometries of $\mathbf{H}^3$ does not hold in general for matrices in $GL_2(\mathsf{C})$. However the formula for Möbius transformations of $\overline{\mathsf{C}}$ holds for every matrix in $GL_2(\mathsf{C})$.

The whole isometry group of $\mathbf{H}^3$ is the semidirect product $PSL_2(\mathsf{C}) \rtimes C_2$, where C_2 is the order two cyclic group whose generator acts on $PSL_2(\mathsf{C})$ by complex conjugation. The generator of C_2 acts on $\mathbf{H}^3 \cup \partial\mathbf{H}^3$ by sending $z + t\,j$ to $\overline{z} + t\,j$, where the bar denotes complex conjugation.

We are interested in isometries which come from $PGL_2(\mathsf{Z}[\omega])$, where $\omega = \exp(2\pi i/3)$, and $\mathsf{Z}[\omega] = \mathsf{Z} + \omega\mathsf{Z}$ is the ring of integers of the number field $\mathsf{Q}(\omega)$. A *unit* of $\mathsf{Z}[\omega]$ is an invertible element of $\mathsf{Z}[\omega]$.

6.2 Lemma. *The identification* $PGL_2(\mathsf{C}) = PSL_2(\mathsf{C})$ *identifies*

$$PGL_2(\mathsf{Z}[\omega]) = PSL_2(\mathsf{Z}[\omega]) \cup PSL_2(\mathsf{Z}[\omega]) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Proof. Let $A \in GL_2(\mathsf{Z}[\omega])$, and let $\lambda \in \mathsf{C}$ be the inverse of a square root of the determinant of A.

The determinant of A is a unit of $\mathsf{Z}[\omega]$, and the group of units of $\mathsf{Z}[\omega]$ is $\{\pm 1, \pm\omega, \pm(1+\omega)\}$, so either λ or λi is a unit of $\mathsf{Z}[\omega]$. Hence λA is contained in either $SL_2(\mathsf{Z}[\omega])$ or $SL_2(\mathsf{Z}[\omega]) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, and the lemma is proved.

Let m, β be two complex numbers, where $m \neq 0$. We write $m\overline{\mathsf{R}} + \beta$ to denote the completed real line $\{mt + \beta \in \mathsf{C} \mid t \in \mathsf{R}\} \cup \{\infty\}$.

6.3 Lemma. *The isometry* $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathsf{C})$ *maps the completed line* $m\overline{\mathsf{R}} + \beta$ *to either a completed line, or a circle in* C *with center*

$$\left(a(\overline{d + \beta c})m - \overline{c}(b + \beta a)\overline{m}\right) / \left(c(\overline{d + \beta c})m - \overline{c}(d + \beta c)\overline{m}\right),$$

and radius

$$|m| / \left|c(\overline{d + \beta c})m - \overline{c}(d + \beta c)\overline{m}\right|.$$

Proof. Since $m\overline{\mathsf{R}} + \beta$ is a circle in the sphere $\overline{\mathsf{C}}$ passing through ∞, it is mapped to either a completed real line, or a circle in C. To prove the formula we may assume that $m = 1$ and $\beta = 0$, by composing $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\pm \begin{pmatrix} \sqrt{m} & \beta/\sqrt{m} \\ 0 & 1/\sqrt{m} \end{pmatrix}$, which sends $\overline{\mathsf{R}}$ to $m\overline{\mathsf{R}} + \beta$. Since $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ maps $-\frac{c}{d}$ to ∞, we may assume that $-\frac{c}{d} \notin \overline{\mathsf{R}}$, so $c\overline{d} - \overline{c}d \neq 0$. The desired result now follows from the fact that, for every $\lambda \in \overline{\mathsf{R}}$,

$$\frac{a\overline{d} - \overline{c}b}{c\overline{d} - \overline{c}d} - \frac{a\lambda + b}{c\lambda + d} = \frac{\overline{c}\lambda + \overline{d}}{c\lambda + d}\, \frac{1}{c\overline{d} - \overline{c}d}, \text{ and } \left|\frac{(\overline{c}\lambda + \overline{d})}{(c\lambda + d)}\right| = 1.$$

6.4 Corollary. *If* $A \in PGL_2(\mathsf{Z}[\omega])$, *and* $\beta, m \in \mathsf{Z}[\omega]$, *and* m *is a unit of* $\mathsf{Z}[\omega]$, *then* A *maps the completed line* $m\overline{\mathsf{R}} + \beta$ *to either a completed line or a circle of radius* r *such that* $r^{-2} \in \mathsf{Z}$.

The next result gives a criterion for a tree of subsets to be contracting.

6.5 Corollary. *Let R be a completed half-plane of $\overline{\mathsf{C}}$ bounded by $m\overline{\mathsf{R}}+\beta$, where $\beta \in \mathsf{Z}[\omega]$, and m is a unit of $\mathsf{Z}[\omega]$. If $(g_n)_{n\in\mathsf{N}}$ is a sequence in $PGL_2(\mathsf{Z}[\omega]) \rtimes C_2$, such that the sequence of sets g_nR is strictly decreasing, then, for any Riemannian metric on $\overline{\mathsf{C}}$, the sequence of the diameters of the g_nR tends to zero.*

Proof. Since $\overline{\mathsf{C}}$ is compact, all the Riemannian metrics on $\overline{\mathsf{C}}$ are equivalent. So, by applying an element of $PGL_2(\mathsf{Z}[\omega])$, we may assume that g_1R is a bounded disc in C. If d_n denotes the diameter of the disc g_nR, then $d_n^{-2} \in \mathsf{Z}$, by Corollary 6.4. Moreover, by hypothesis, the d_n form a strictly decreasing sequence, which must tend to zero.

In order to apply results about $\partial\mathbf{H}^3$ to $\mathbf{H}^3$, we will use convex hulls, as we did in the two-dimensional case. We record the three-dimensional version of Lemma 3.5.

6.6 Lemma. *Let $(D_n)_{n\in\mathsf{N}}$ be a decreasing sequence of discs in $\overline{\mathsf{C}}$, such that the sequence of their diameters tends to zero, and let $\{x_\infty\}$ be the intersection of all the D_n. If $(x_n)_{n\in\mathsf{N}}$ is a sequence in $\mathbf{H}^3$ such that, for each $n \in \mathsf{N}$, x_n is contained in the convex hull of D_n, then the sequence of x_n tends to x_∞.*

Proof. We may assume that the discs are contained in C. Thus, the convex hull of each D_n is a half-ball having the same (Euclidean) diameter as D_n. In particular, the sequence of diameters of the convex hulls tends to zero.

7. The action of G on $\mathbf{H}^3 \cup \partial\mathbf{H}^3$

In this section we describe the action of G on $\mathbf{H}^3 \cup \partial\mathbf{H}^3$, and we introduce its normalizer in the isometry group of $\mathbf{H}^3$.

7.1. Recall the action of G on $\mathbf{H}^3$ defined in the introduction. This action is given by

$$a \mapsto \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad b \mapsto \pm\begin{pmatrix} -w & w \\ 1 & w \end{pmatrix}, \qquad \text{and} \qquad c \mapsto \pm\begin{pmatrix} w & 1 \\ w & -w \end{pmatrix}.$$

Since the images belong to $PSL_2(\mathsf{Z}[\omega])$, this action is discrete.

We will show, in Proposition 8.11, that the action of G is faithful, and, in Section 11, that the restriction of this action to the index two subgroup (freely) generated by ba and bc is the action of the fundamental group of the fibre of the Gieseking manifold. This action was studied by Jørgensen and Marden in [JM], who showed, in particular, that it is discrete and faithful.

7.2 Definitions. Let $\mathrm{Aut}(G)$ denote the group of automorphisms of G. Since the centre of G is trivial, we view G as a subgroup of $\mathrm{Aut}(G)$, by identifying each element of G with left conjugation by that element. Thus G is a normal subgroup of $\mathrm{Aut}(G)$, and the elements of the latter act on the elements of the former by left conjugation.

Let $d, e \in \mathrm{Aut}(G)$ be defined by

$$d\colon (a,b,c) \mapsto (a,c,b), \qquad e\colon (a,b,c) \mapsto (b, cac, c).$$

Thus

(4) $dad^{-1} = a$, $dbd^{-1} = c$, $dcd^{-1} = b$,
(5) $eae^{-1} = b$, $ebe^{-1} = cac$, $ece^{-1} = c$.

Let $\widehat{G}$ denote the subgroup of $\mathrm{Aut}(G)$ generated by d, e. Direct calculation shows that

(6) $a = e^{-1}de^2de$,
(7) $b = de^2d$,
(8) $c = e^2$,

so G lies in $\widehat{G}$. Moreover,

(9) $d^2 = e^4 = (e^3de^2ded)^2 = 1$.

In fact $\widehat{G}$ can be identified with the group

$$P = \langle d, e \mid d^2 = e^4 = (e^3de^2ded)^2 = 1\rangle.$$

To see this, notice that (6),(7),(8) define a homomorphism from G to P, and then (4), (5) are consequences of the defining relations $d^2 = e^4 = (ad)^2 = 1$. Thus G is a normal subgroup of P, and $P/G \cong C_2 * C_2$. In order to be able to identify $P = \widehat{G}$, we need only show that the surjective map from P/G to $\widehat{G}/G$ is injective. Since every proper quotient of $C_2 * C_2$ is finite, it suffices to show that $\widehat{G}/G$ is infinite. But this is clear, since no proper power of $de\colon (ba, bc) \mapsto (babc, ba)$ acts trivially on the free group abelianized. Thus

$$\widehat{G} = \langle d, e \mid d^2 = e^4 = (e^3de^2ded)^2 = 1\rangle.$$

Now $\mathrm{Aut}(G)$ acts on the vertices of the tree $\mathcal{T}_G$, and by a standard bounded cancellation argument, this action extends to a continuous action on the set $\partial\mathcal{T}_G$ of right infinite reduced words. We shall be interested in the subgroup $\widehat{G}$ of $\mathrm{Aut}(G)$, and here it is easy to verify directly that $\widehat{G}$ has a well-defined action on the set $\partial\mathcal{T}_G$ of right infinite reduced words. We record the following aspects of the action, which make continuity obvious.

$(10')\ d[a] = [a].$	$(13')\ e[a] = [b].$
$(11')\ d[b] = [c].$	$(14')\ e[b] = c[a].$
$(12')\ d[c] = [b].$	$(15')\ e[c] = [a] \cup c[b].$

Here $(10')$, $(11')$, $(12')$, and $(13')$ follow directly from (4) and (5). To see $(14')$, notice that $c[a] = e^2[a] = e[b]$, by (8) and $(13')$. Finally, $(15')$ holds since

$$e[c] = ec[a] \cup ec[b] = ce[a] \cup ce[b] = c[b] \cup [a].$$

7.3 Lemma. *For every vertex $v \in V(\mathcal{T}_G) = G$, and every $\gamma \in \widehat{G}$, there exist $v_1, \dots, v_k \in V(\mathcal{T}_G)$ such that $\gamma[v] = [v_1] \cup \cdots \cup [v_k]$.*

7.4 Definition. Let $\widehat{G}$ act as Möbius transformations on $\overline{\mathsf{C}}$ by setting

$$d(z) = \frac{1}{z}, \qquad \text{and} \qquad e(z) = \frac{1}{\overline{\omega z} + 1} = \frac{1}{(1+\omega)\overline{z} + 1},$$

for every $z \in \overline{\mathsf{C}}$. Direct calculation shows that (9) is respected, and (6)–(8) give the action recalled in Subsection 7.1. Thus we have extended the G-action on $\mathbf{H}^3$ to a $\widehat{G}$-action. Once we have shown that the G-action is faithful, it will be a simple matter to show that the $\widehat{G}$-action is faithful, and that the image of $\widehat{G}$ is the whole normalizer of the image of G in $PSL_2(\mathsf{C})$ o C_2; see Proposition 8.11.

FIGURE 8.1. The region S.

FIGURE 8.2. The curves L, dL, edL, and $dedL$.

8. THE BREAKUP OF $\overline{\mathsf{C}}$

Proof of Theorem D. The proof will be based on the following result.

8.1 Lemma. *Let S be the region of $\overline{\mathsf{C}}$ bounded by $\omega\overline{\mathsf{R}_+}$ and $(1+\omega)\overline{\mathsf{R}_+}$ and containing i, where $\overline{\mathsf{R}_+} = [0,\infty]$, as in Figure 8.1. There exists a simple curve L from 0 to ∞ contained in S such that $L = aL$ and $L = e^{-1}dL \cup e^{-1}dedL$.*

We will prove Lemma 8.1 at the end of this section, and we now proceed to prove Theorem D assuming Lemma 8.1.

We consider the curves L, dL, edL, and $dedL$. The next result tells us that these curves are arranged in $\overline{\mathsf{C}}$ as suggested by Figure 8.2.

8.2 Lemma. (i) $L \cap dL = \{0, \infty\}$.
(ii) $L \cap edL = \{0\}$.
(iii) $L \cap dedL = \{\infty\}$.
(iv) $dL \cap edL = \{0\}$.
(v) $dL \cap dedL = \{\infty\}$.
(vi) $edL \cap dedL = \{1\}$.

All these intersections occur at the end-points of the curves.

Proof of Lemma 8.2. Since the end-points of L are 0 and ∞, and since $d(0) = \infty$, $d(\infty) = 0$, $ed(0) = 0$, $ed(\infty) = 1$, $ded(0) = \infty$, and $ded(\infty) = 1$, it remains to prove that the intersections of the curves are contained in the claimed sets. To show this, we prove that (i)-(vi) all hold when we replace L by S; see

Figure 8.3. For example, dS is the region bounded by $-\omega\overline{\mathsf{R}_+}$ and $-(1+\omega)\overline{\mathsf{R}_+}$ containing $-i$, and (i) is easily verified. By using Lemma 6.3, we easily check that edS is the compact region of C which is bounded by the segment $[0,1]$, and an arc of the circle which passes through 0 and 1 and is tangent to the lines $(1+\omega)\mathsf{R}$ and $1+\omega\mathsf{R}$. Moreover $dedS$ is the region bounded by $1+(1+\omega)\overline{\mathsf{R}_+}$ and $1-\omega\overline{\mathsf{R}_+}$, and contains 2. From these computations, (i)–(vi) follow easily.

Figure 8.3. The regions S, dS, edS, and $dedS$.

8.3. We return to the proof of Theorem D, and define the regions R_a, R_b, and R_c as follows. Let R_a be the region bounded by $L \cup dL$ not containing edL; let R_b be the region bounded by $dL \cup edL \cup dedL$ not containing L; and, let R_c be the region bounded by $L \cup edL \cup dedL$ not containing dL. See Figure 8.2. It follows from Lemma 8.2, and these definitions, that R_a, R_b, and R_c form a Jordan partition of C.

Thus we have seen (0), and we now check (1), (2), and (3).

Since $aL = L$ and $ad = da$, a preserves the boundary of R_a. Moreover, as $a(0) = \infty$ and $a(\infty) = 0$, a reverses the orientation of the curve L and the orientation of ∂R_a. This proves (1), because a preserves the orientation of $\overline{\mathsf{C}}$.

Similarly, in order to prove (2), we show that b preserves ∂R_b and reverses the orientation of ∂R_b. Since $b = eae^{-1}$, it suffices to prove that $e\partial R_a = \partial R_b$. By Lemma 8.1, $eL = dL \cup dedL$. Hence,

$$e\partial R_a = eL \cup edL = dL \cup dedL \cup edL = \partial R_b.$$

Similarly, to prove (3), since $c = dbd$, it suffices to check that $d\partial R_b = \partial R_c$, and this is straightforward.

FIGURE 8.4. The region X.

8.4. It remains to prove Lemma 8.1.

Proof of Lemma 8.1. We map the monoid B to $\widehat{G}$ by $f \mapsto e^{-1}d$ and $g \mapsto e^{-1}ded$. By Definition 7.4, we then have a B-action on $\overline{\mathsf{C}}$ such that

$$f(z) = \overline{(\omega z - \omega)} \quad \text{and} \quad g(z) = -1/(z+\omega),$$

for all $z \in \overline{\mathsf{C}}$.

Let $X \subseteq S$ be the region bounded by $w\overline{\mathsf{R}_+}$, $(1+\omega)+w\overline{\mathsf{R}_+}$, and the segment $[0, 1+\omega]$. It is easily checked that $fX \subset X$, and $gX \subset X$, so we can consider the restriction of the action of B to X. See Figure 8.4.

Here there is a B-equivariant tree of subsets $\sigma : V(\mathcal{T}_B) \to \mathcal{P}(X)$, defined by $\sigma(v) = v(X)$, for all $v \in V(\mathcal{T}_B) = B$, as in Definition 2.6. We will prove, in Subsection 8.5, that this tree is contracting. Hence there exists a continuous B-equivariant map $\tilde{\eta} : \partial\mathcal{T}_B \to X$. In Subsection 8.6, we show that $\tilde{\eta}$ factors through $\nu : \partial\mathcal{T}_B \to [0, +\infty]$ to give a map $\eta : [0, +\infty] \to X$ such that $\eta(0) = 0$ and $\eta(\infty) = \infty$. In Lemma 8.7 and Subsection 8.8, we show that η is injective, and in Subsection 8.9 we show that $L = \eta([0, +\infty])$ satisfies the conclusions of Lemma 8.1.

8.5. In order to prove that σ is contracting, we view every right infinite word in f and g as a right infinite word in g, fg, and f^2. Let Y be the completed

FIGURE 8.5. The region Y.

half-plane $Y = \{z \in \mathsf{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$. Then gY, fgY, and f^2Y are strictly contained in Y, as depicted in Figure 8.5.

Let $w = w_1 w_2 \cdots$ be a right infinite word, where $w_n \in \{g, fg, f^2\}$ for each $n \in \mathsf{N}$. The sequence of subsets $(w_1 w_2 \cdots w_n Y)_{n \in \mathsf{N}}$ is a strictly decreasing sequence of discs. Thus, by Corollary 6.5, the sequence of diameters of the $w_1 w_2 \cdots w_n Y$ tends to zero. Since $X \subseteq Y$, it follows that the tree σ is contracting and, by Proposition 2.4, we get a continuous B-equivariant map $\tilde{\eta}\colon \partial \mathcal{T}_B \to X$.

8.6. In order to prove that $\tilde{\eta}$ factors through ν to give a map $\eta\colon [0, \infty] \to X$, we show that $\tilde{\eta}(g^2 f^\infty) = \tilde{\eta}(fgf^\infty)$, and apply Proposition 5.4. Since $\tilde{\eta}$ is B-equivariant, $\eta(f^\infty) = \infty$, because ∞ is the only point of $\overline{\mathsf{C}}$ fixed by f. So

$$\tilde{\eta}(g^2 f^\infty) = g^2(\infty) = 1 + \omega \qquad \text{and} \qquad \tilde{\eta}(fgf^\infty) = fg(\infty) = 1 + \omega.$$

Moreover, since $\nu(f^\infty) = \infty$, and $\nu(gf^\infty) = 0$, we see that $\eta(\infty) = \tilde{\eta}(f^\infty) = \infty$, and $\eta(0) = \tilde{\eta}(gf^\infty) = g(\infty) = 0$, as claimed.

This gives us a B-equivariant map $\eta\colon [0, \infty] \to X$, with $\eta(0) = 0$. Hence the B-orbit of the 0 in $[0, \infty]$ maps to the B-orbit of the 0 in X, and this gives a recursive formula for η on the non-negative rational numbers. Thus we can calculate η explicitly on a reasonable finite set of non-negative rationals, and then join the dots to obtain detailed approximations of the image of η, as in Figure 8.6.

FIGURE 8.6 The image L of η with the dots connected in the right order. CALVIN AND HOBBES copyright Watterson. Reprinted with permission of UNIVERSAL PRESS SYNDICATE. All rights reserved.

We will use the following to prove the injectivity of η.

8.7 Lemma. *$fX \cap gX = \{\eta(1)\} = \{1+\omega\}$, and, for all $n \geq 0$, $gX \cap f^2X = gX \cap fgf^ngX = \emptyset$, and $fX \cap gfX = fX \cap g^2f^ngX = \emptyset$.*

Proof. It is easily checked that $fX \subseteq \{z \in \mathsf{C} \mid \mathrm{Im}(z) \geq \sqrt{3}/2\} \cup \{\infty\}$ and gX is contained in the circle $g(1+\omega)\overline{\mathsf{R}}$, which has center $(2+\omega)/3$ and radius $\sqrt{3}/3$. It follows that $1+\omega$ is the only possible point of the intersection $fX \cap gX$. Since $f(0) = g(0) = 1+\omega$, we conclude that $fX \cap gX = \{1+\omega\}$.

From the foregoing descriptions of fX and gX, we see that $0 \notin fX$, and $\infty \notin gX$. In particular, $1+\omega = f(0)$ is not contained in f^2X, and thus f^2X does not meet gX. Since $\infty \notin gX$, and $f(\infty) = \infty$, we see that $1+\omega = fg(\infty)$ is not contained in fgf^ngX, so $gX \cap fgf^ngX = \emptyset$.

A similar argument shows $fX \cap gfX = fX \cap g^2f^ngX = \emptyset$.

8.8. For the B-action on $[0,\infty]$ described in Notation 5.1, we have the following.

$$\begin{aligned}
(1,\infty] &= f[0,\infty] - \{1\} = f^2[0,\infty] \cup \bigcup_{n>0} fgf^ng[0,\infty],\\
[0,1) &= g[0,\infty] - \{1\} = gf[0,\infty] \cup \bigcup_{n>0} g^2f^ng[0,\infty],\\
\text{and} \qquad \{1\} &= f[0,\infty] \cap g[0,\infty].
\end{aligned}$$

Now Lemma 8.7 shows that for any $x \in [0,1)$, and $y \in (1,\infty]$, $\eta(x)$, $\eta(1)$, and $\eta(y)$ are distinct.

Consider any $x \neq y$ in $[0,\infty]$. There exists a longest word $w \in B$ such that $x, y \in w[0,\infty]$. Let $x', y' \in [0,\infty]$ be such that $x = w(x')$ and $y = w(y')$. By maximality of w, either $x' \in [0,1]$ and $y' \in (1,\infty]$, or $y' \in [0,1]$ and $x' \in (1,\infty]$, and we may assume the former. Thus $\eta(x') \neq \eta(y')$ and, since η is B-equivariant, and w acts as a Möbius transformation, we deduce that $\eta(x) \neq \eta(y)$. This proves the injectivity of η.

8.9. We define L to be $\eta[0,\infty]$.

To show that $L = aL$, we view each point of L as the intersection of a decreasing sequence of compact subsets of the form w_nY, where w_n is a word in f^2, fg, and g, and $Y = \{z \in \mathsf{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$, as in Subsection 8.5. By (4) and (6), $af^2 = g$, $ag = f^2$, and $afg = fga$. Moreover $aY = Y$. So, for each $n > 0$, $aw_nY = w'_nY$ for some word w'_n in f^2, g, and fg. Thus $aL = L$.

Finally, since $[0,\infty] = f[0,\infty] \cup g[0,\infty]$, and η is B-equivariant, we have $L = fL \cup gL$.

This finishes the proofs of Lemma 8.1 and Theorem D.

By Theorem D, we have a tree of subsets $\sigma : \mathcal{T}_G \to \mathcal{P}(\overline{\mathbb{C}})$, as in Definition 2.7. (Let us make the whimsical remark that we have a degenerate Schottky group, since the Möbius transformations ab and ac carry the interiors of the Jordan regions R_b and R_c to the exteriors of the Jordan regions aR_b and aR_c, respectively. Of course, this is not a Shottky group since its limit set is the whole Riemann sphere; the crucial condition which fails to be satisfied is that the Jordan curves in question be disjoint.)

We will use the following property of σ in the proof of Theorem C.

8.10 Lemma. *If $v \in V(\mathcal{T}_G) = G$, and $\gamma \in \widehat{G}$, then $\gamma\sigma(v) = \sigma(v_1) \cup \cdots \cup \sigma(v_k)$ for any $v_1, \ldots, v_k \in V(\mathcal{T}_G) = G$ such that $\gamma[v] = [v_1] \cup \cdots \cup [v_k]$.*

Proof. We claim that the following hold.

$$
\begin{array}{ll}
(10)\, dR_a = R_a. & (13)\, eR_a = R_b. \\
(11)\, dR_b = R_c. & (14)\, eR_b = cR_a. \\
(12)\, dR_c = R_b. & (15)\, eR_c = R_a \cup cR_b.
\end{array}
$$

Since $\partial R_a = L \cup dL$, and $d^2 = 1$, d preserves ∂R_a. Moreover -1 is in the interior of R_a, and is fixed by d, so (10) holds. Also (11) and (12) follow from (10), and the fact that $d\partial R_b = \partial R_c$, proved in Subsection 8.3. In Subsection 8.3, we also showed that $e\partial R_a = \partial R_b$. Since $e(0) = 1$, $e(1+\omega) = \frac{1}{2}$, $e(\infty) = 0$, and e is orientation-reversing, we obtain (13). Now (14) follows from (13), since $c = e^2$. By combining (3), (13), and (14), we get (15), since $c = e^2 = e^{-2}$.

The lemma now follows from (10′)-(15′), and (10)-(15).

8.11 Proposition. *In $PSL_2(\mathbb{C}) \rtimes C_2$, the (faithful) image of $\widehat{G}$ is the normalizer of the (faithful) image of G.*

Proof. It follows from Theorem D that G acts faithfully on $\overline{\mathbb{C}}$. Since $\widehat{G}$ is a subgroup of $\mathrm{Aut}(G)$, it follows that $\widehat{G}$ acts faithfully on $\overline{\mathbb{C}}$. For the purposes of this proof we shall identify elements of $\widehat{G}$ with their images in $PSL_2(\mathbb{C}) \rtimes C_2$.

Suppose that $h \in PSL_2(\mathbb{C}) \rtimes C_2$ normalizes G, so left conjugation by h induces an automorphism of G. We wish to show that h lies in $\widehat{G}$. By replacing h with he, if necessary, we may assume that $h \in PSL_2(\mathbb{C})$.

Consider the subgroup G_0 of G generated by $x = ab$ and $y = ca$. Thus G is the semidirect product of G_0 with the subgroup of order two generated by a, and G_0 is free of rank two. Since G_0 is the only free subgroup of index two in G, we see that h induces an automorphism of G_0. Let $z = xyx^{-1}y^{-1} = abcabc$. By a classic group-theoretic result of Nielsen, every automorphism of G_0 carries z to a conjugate of z or z^{-1}. But $dzd^{-1} = y^{-1}x^{-1}z^{-1}xy$, so, on replacing h with hd, if necessary, we may assume that $hzh^{-1} = gzg^{-1}$ for some $g \in G$. On replacing h with $g^{-1}h$, we may assume that h commutes with $z = \pm\begin{pmatrix} 1 & -4-2\omega \\ 0 & 1 \end{pmatrix}$.

Since $h \in PSL_2(\mathsf{C})$, it follows that $h = \pm\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ for some $r \in \mathsf{C}$. Then

$$\pm\begin{pmatrix} -r & r^2+1 \\ -1 & r \end{pmatrix} = hah^{-1} \in G \leq PSL_2(\mathsf{Z}[\omega]),$$

so $r \in \mathsf{Z}[\omega]$. Since $\pm\begin{pmatrix} 1 & -\omega \\ 0 & 1 \end{pmatrix} = dede \in \widehat{G}$, we may assume that $r \in \mathsf{Z}$. Also $\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = cbadede \in \widehat{G}$, so we may assume that $r = 0$ or 1.

It suffices to show that $r = 1$ is impossible. This can be seen by working in $\mathsf{Z}[\omega]$ modulo 2, since a straightforward calculation shows that the resulting image of G has order 10, and contains no elements with 1s on the diagonal. Alternatively, we can argue as follows. Assume $h = \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so $h^2 = cbadede$. Here

$$h^2\colon (x, y) \mapsto (x^{-3}y^{-1}, yxy^{-1}),$$

and on the abelianization of G_0, h^2 acts as the matrix $\begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}$. Hence this matrix is the square of a 2×2 integer matrix, but again, a straightforward calculation shows this is impossible, as desired.

9. The map $\tilde{\Phi} : \partial\mathcal{T}_G \to \partial\mathbf{H}^3$.

In this section we prove Theorem C and Corollary E, and then, in Subsection 9.10, we describe approximations of $\tilde{\Phi}$.

Proof of Theorem C. To construct $\tilde{\Phi}$ we use the regions R_a, R_b, and R_c given by Theorem D, and the resulting tree of subsets $\sigma\colon \mathcal{T}_G \to \mathcal{P}(\overline{\mathsf{C}})$ of Definition 2.7. Proposition 9.5 will show that σ is contracting. In Lemma 9.6, we show this gives a G-equivariant map $\tilde{\Phi}$, by Proposition 2.4, and $\tilde{\Phi}$ is the continuous extension of any G-equivariant map $j\colon \mathcal{T}_G \to \partial\mathbf{H}^3$. In Subsection 9.8, we prove that $\tilde{\Phi}$ factors through τ.

9.1 Notation. In $\widehat{G}$, we identify $f = e^{-1}d$, as in Subsection 8.4, and we let

$$w_1 = ca, \quad w_2 = cad, \quad w_3 = cbaf^{-1}cad, \quad w_4 = cbaf^{-1}ca, \quad w_5 = cbaf^{-2},$$

$$\mathcal{W} = \{w_1, w_2, w_3, w_4, w_5\} \cup \bigcup_{n\geq 1} f^{2n}\{w_1, w_2, w_3, w_4\} \cup \bigcup_{n\geq 1} f^{-2n}\{w_4, w_5, w_5w_1, w_5w_2\}.$$

FIGURE 9.1. R, w_1R, w_2R, w_3R, w_4R, and w_5R.

In $\partial\mathcal{T}_G$, let $w_{+\infty} = \lim\limits_{n\to+\infty} f^{2n}cbcf^{-2n}$, and $w_{-\infty} = \lim\limits_{n\to+\infty} f^{-2n}cbabf^{2n}$, so

$$w_{+\infty} = cbcacbcacacbc\cdots, \qquad w_{-\infty} = cbababcbababc\cdots.$$

9.2 Proposition. *Every element of $[c] \subset \partial\mathcal{T}_G$ is either a right infinite word in $\mathcal{W}$, or a (finite) word in $\mathcal{W}$, followed by an infinite tail $w_{+\infty}$ or $w_{-\infty}$.*

Proof. By (7′)–(15′), $c[a] = w_1[c] \cup w_2[c]$, $cbab[c] = w_3[c] \cup w_4[c]$, and

$$cba[c] \cup cbabab[a] \cup cbaba[c] = w_5[c],$$

so $[c] = \bigcup\limits_{n=1}^{5} w_n[c] \cup cb[c] \cup cbabab[c]$, and it remains to decompose $cb[c]$ and $cbabab[c]$. Since

$$f^{-2}cb[c] = cbab[c] \cup c[a] \cup cb[c] = \bigcup_{n=1}^{4} w_n[c] \cup cb[c],$$

it follows that every element of $cb[c] - \{w_{+\infty}\}$ belongs to $f^{2n}w_m[c]$ for some $n \geq 1$, and some $m \in \{1,2,3,4\}$.

Similarly, since

$$f^2cbabab[c] = cba[b] = w_3[c] \cup w_4[c] \cup w_5w_1[c] \cup w_5w_2[c] \cup cbabab[c],$$

every element of $cbabab[c] - \{w_{-\infty}\}$ lies in $w[c]$ for some $w \in \mathcal{W}$.

Consider the completed half-plane R bounded by the completed line $\omega\overline{\mathsf{R}}$ and containing 1, as in Figure 9.1.

FIGURE 9.2. The region R'.

9.3 Lemma. *For the $\widehat{G}$-action on $\overline{\mathsf{C}}$ of Definition 7.4, the following hold.*

(i) *For every $w \in \mathcal{W}$, $wR \subset R$.*
(ii) *For every $n > 0$, $f^{2n}cbR_c \subseteq \{z \in \mathsf{C} \mid \mathrm{Im}(z) \geq n\sqrt{3}/2\} \cup \{\infty\}$, a completed half-plane.*
(iii) *For every $n > 0$, $f^{-2n}cbaR_b \subseteq \{z \in \mathsf{C} \mid \mathrm{Im}(z) \leq -n\sqrt{3}/2\} \cup \{\infty\}$, a completed half-plane.*

Proof. (i) Since $f^2R = R$, it suffices to prove $w_nR \subset R$ for $n = 1, 2, 3, 4, 5$. Since $w_5(z) = z + 2$, it is clear that $w_5R \subset R$. For $n = 1, 2, 3, 4$, $w_n(R)$ is a bounded disc in C, and it is easily checked that $w_n(R) \subset R$ by Lemma 6.3; see Figure 9.1. This completes the proof of (i).

(ii) Consider the region $R' \subset R$, which contains 2, and is bounded by $\omega\overline{\mathsf{R}_+}$ and $[0, 1]$ and $1 - \omega\overline{\mathsf{R}_+}$, as in Figure 9.2. Then $R_c \subseteq R'$; see Figures 8.2 and 8.3. It is easily verified that

$$cbR' \subseteq \{z \in \mathsf{C} \mid \mathrm{Im}(z) \geq 0\} \cup \{\infty\},$$

a completed half-plane. Now (ii) follows from the fact that $f^2(z) = z + \omega$.

(iii) We have $R_b = dR_c \subset dR'$, and it is easily checked that

$$dR' \subseteq \{z \in \mathsf{C} \mid \mathrm{Im}(z) \leq \sqrt{3}/2\} \cup \{\infty\}.$$

Since $cba(z) = z + 2 + \omega$, and $f^{-2}(z) = z - \omega$, (iii) holds.

9.4 Corollary. *With respect to any Riemannian metric on $\overline{\mathsf{C}}$, for any sequence of elements γ_n of $\mathcal{W}$, the sequence of the diameters of the $\gamma_1 \cdots \gamma_n R$ tends to zero.*

Proof. By the previous lemma, $(\gamma_1 \cdots \gamma_n(R))_{n \in \mathsf{N}}$ is a sequence of strictly decreasing subsets, so Corollary 6.5 applies.

9.5 Proposition. *The regions R_a, R_b, and R_c of Theorem D, determine a tree of subsets $\sigma : V(\mathcal{T}_G) \to \mathcal{P}(\overline{\mathsf{C}})$, by Definition 2.7. This tree of subsets is contracting.*

Proof. In $\partial\mathcal{T}_G$, let $\mathsf{e} = \alpha_1\alpha_2\cdots$ be a right infinite reduced word with each $\alpha_n \in \{a, b, c\}$. We want to show that the sequence of diameters of the $\sigma(\alpha_1\cdots\alpha_n)$ tends to zero.

If α_1 is a or b, then, since c is a Möbius transformation, we can replace e with $c\mathsf{e} = c\alpha_1\alpha_2\cdots$; thus we may assume that $\alpha_1 = c$.

Consider first the case where e is a right infinite word in $\mathcal{W}$, so $\mathsf{e} = \gamma_1\gamma_2\cdots$, where each $\gamma_n \in \mathcal{W}$.

By Lemma 7.3, for each $n > 0$, there exist $v_1, \ldots, v_k \in G$ such that

$$\gamma_1\cdots\gamma_n[c] = [v_1] \cup \cdots \cup [v_k].$$

Since $\mathsf{e} \in \gamma_1\cdots\gamma_n[c]$, one of the open sets $[v_1], \ldots, [v_k]$, contains e, say $\mathsf{e} \in [v_1]$. Thus there exists m_n such that $\alpha_1\cdots\alpha_{m_n} = v_1$. Moreover, by Lemma 8.10, $\gamma_1\cdots\gamma_n R_c = \sigma(v_1) \cup \cdots \cup \sigma(v_k)$. We conclude that $\sigma(\alpha_1\cdots\alpha_{m_n}) = \sigma(v_1) \subseteq \gamma_1\cdots\gamma_n R_c$. By Corollary 9.4, the sequence of diameters of the $\sigma(\alpha_1\cdots\alpha_{m_n})$ tends to zero. So the sequence of diameters of the $\sigma(\alpha_1\cdots\alpha_n)$ also tends to zero.

Consider next the case where $\mathsf{e} = c\alpha_2\cdots$ is the product of a (finite) word in $\mathcal{W}$ and the tail $\omega_{+\infty}$. Since every element of $\mathcal{W}$ acts as a Möbius transformation, and all Riemannian metrics on $\overline{\mathsf{C}}$ are equivalent, we may assume that $\mathsf{e} = \omega_{+\infty}$. Since $\omega_{+\infty} \in cb[c]$ and $f\omega_{+\infty} = \omega_{+\infty}$, we see that $\omega_{+\infty} \in f^{2n}cb[c]$. By an argument similar to the previous case, there is an increasing sequence of natural numbers m_n such that, for every $n > 0$, $\sigma(\alpha_1\cdots\alpha_{m_n}) \subseteq f^{2n}cbR_c$. By Lemma 9.3 (ii), the sequence of diameters of the $\sigma(\alpha_1\cdots\alpha_{m_n})$ tends to zero.

By Proposition 9.2, we are left with the case where e ends with $\omega_{-\infty}$, where the argument is similar to the preceding case.

9.6 Lemma. *The contracting tree of subsets $\sigma : V(\mathcal{T}_G) \to \mathcal{P}(\overline{\mathsf{C}})$ of Proposition 9.5, determines a G-equivariant quotient map $\tilde{\Phi} : \partial\mathcal{T}_G \to \overline{\mathsf{C}} = \partial\mathbf{H}^3$, as in Proposition 2.4. This map is the continuous extension of any G-equivariant map $j : \mathcal{T}_G \to \mathbf{H}^3$.*

Proof. Let $(v_n)_{n\in\mathsf{N}}$ be a sequence of vertices in $\mathcal{T}_G$ converging to an end $\mathsf{e} \in \partial\mathcal{T}_G$. We want to show that the sequence of $j(v_n)$ tends to $\tilde{\Phi}(\mathsf{e})$.

By G-equivariance, $j(v_n) = v_n j(1)$, for every $n > 0$. Moreover, since the convergence, and the limit of the sequence of $v_n j(1)$, are independent of the point $j(1)$, we may assume that $j(1)$ belongs to the geodesic with ends 0 and ∞, so $j(1)$ is contained in the convex hull of $R_a \cap R_b \cap R_c$. Hence, for each

n, $j(v_n)$ belongs to the convex hull of $\sigma(v_n)$. Since the sequence of v_n tends to e, and σ is contracting, the sequence of $j(v_n)$ tends to the unique element of $\bigcap_{n>0} \sigma(v_n) = \{\tilde{\Phi}(\mathsf{e})\}$, by Lemma 6.6. This proves that $\tilde{\Phi}$ is the continuous extension of j.

9.7 Corollary. *The map $\tilde{\Phi}\colon\partial\mathcal{T}_G \to \overline{\mathsf{C}}$ is $\widehat{G}$-equivariant.*

Proof. Let $\mathsf{e} \in \partial\mathcal{T}_G$ and $\gamma \in \widehat{G}$. It suffices to show that $\tilde{\Phi}(\gamma\mathsf{e}) = \gamma\tilde{\Phi}(e)$.

Fix a point $x \in \mathbf{H}^3$, let $y = \gamma^{-1}x \in \mathbf{H}^3$, and let $v_0, v_1, \dots$ be a ray in $\mathcal{T}_G$ representing the end e, so $\mathsf{e} = \lim_{n\to\infty} v_n$. Since γ acts continuously on $\mathcal{T}_G \cup \partial\mathcal{T}_G$, $\gamma\mathsf{e} = \lim_{n\to\infty} \gamma v_n \gamma^{-1}$. By Lemma 9.6, $\tilde{\Phi}(\gamma\mathsf{e}) = \lim_{n\to\infty} \gamma v_n \gamma^{-1} x$, that is, $\lim_{n\to\infty} \gamma v_n y$. Since γ acts continuously on $\mathbf{H}^3 \cup \partial\mathbf{H}^3$, $\lim_{n\to\infty} \gamma v_n y = \gamma \lim_{n\to\infty} v_n y$. By Lemma 9.6 again, $\lim_{n\to\infty} v_n y = \tilde{\Phi}(\mathsf{e})$, and we have the desired result.

9.8. Finally, we show that $\tilde{\Phi}\colon\partial\mathcal{T}_G \to \overline{\mathsf{C}}$ factors through $\tau\colon\partial\mathcal{T}_G \to \overline{\mathsf{R}}$.

Since $abc(z) = z-2-\omega$, ∞ is the only point fixed by abc. By G-equivariance,

$$\tilde{\Phi}((abc)^\infty) = \tilde{\Phi}((cba)^\infty) = \infty,$$

because $(abc)^\infty$ and $(cba)^\infty$ are fixed by abc. Thus, by Proposition 4.5, $\tilde{\Phi}$ has the desired factorization.

This finishes the proof of Theorem C.

9.9. *Proof of Corollary E.* Let p, q, r, s be integers such that $ps - qr = \pm 1$, and, if $qs \neq 0$, $\frac{p}{q} < \frac{r}{s}$. We want to show that $\Phi([\frac{p}{q}, \frac{r}{s}])$ and $\Phi(\overline{\mathsf{R}} - [\frac{p}{q}, \frac{r}{s}])$ form a Jordan partition of $\partial\mathbf{H}^3$.

Since there is a single G-orbit of triangles in the Farey tessellation, there exists $\gamma \in G$ such that $[\frac{p}{q}, \frac{r}{s}]$ is equal to $\gamma[-\infty, 0]$ or $\gamma[0,1]$ or $\gamma[1, +\infty]$.

Consider first the case where $[\frac{p}{q}, \frac{r}{s}] = \gamma[-\infty, 0]$.

Since R_a and $R_b \cup R_c$ form a Jordan partition of $\overline{\mathsf{C}}$, so do

$$\Phi[\frac{p}{q}, \frac{r}{s}] = \Phi\gamma[-\infty, 0] = \gamma R_a,$$

$$\text{and} \qquad \Phi(\overline{\mathsf{R}} - (\frac{p}{q}, \frac{r}{s})) = \Phi\gamma([0,1] \cup [1, \infty]) = \gamma R_b \cup \gamma R_c.$$

Thus it remains to show that $\Phi(\overline{\mathsf{R}} - (\frac{p}{q}, \frac{r}{s})) = \Phi(\overline{\mathsf{R}} - [\frac{p}{q}, \frac{r}{s}])$. By G-equivariance, we may assume that $[\frac{p}{q}, \frac{r}{s}] = [-\infty, 0]$, and it remains to show that $\Phi((0, +\infty)) = \Phi([0, +\infty])$, that is, there exist $x, y \in (0, +\infty)$ such that $\Phi(x) = \Phi(\infty) = \infty$, and $\Phi(y) = \Phi(0) = 0$.

Since $\Phi\tau = \tilde{\Phi}$, it suffices to find ends $\tilde{x} \neq (cba)^\infty$, and $\tilde{y} \neq bc(abc)^\infty$, in $[b] \cup [c]$, such that $\tilde{\Phi}(\tilde{x}) = \infty$, and $\tilde{\Phi}(\tilde{y}) = 0$. We take

$$\tilde{x} = w_{+\infty} = \lim_{n\to+\infty} f^{2n} cbc f^{-2n} = cbcacbcacacbc\cdots,$$

$$\text{and} \qquad \tilde{y} = d\tilde{x} = bcbabcbababcb\cdots.$$

Since $\tilde{\Phi}$ is $\widehat{G}$-equivariant, by Corollary 9.7, and ∞ is the only point of $\overline{\mathsf{C}}$ fixed by f^2, we see $\tilde{\Phi}(\tilde{x}) = \tilde{\Phi}(w_{+\infty}) = \infty$. By the $\widehat{G}$-equivariance of $\tilde{\Phi}$, $\tilde{\Phi}(\tilde{y}) = d(\infty) = 0$.

When $[\frac{p}{q}, \frac{r}{s}]$ is equal to $\gamma[0,1]$, or $\gamma[-\infty, 0]$, the argument is analogous, by permuting R_a, R_b, and R_c, and the points 0, 1, and ∞.

9.10. The results of this section can be used to calculate approximations of Φ.

The tree of subsets of $\partial\mathcal{T}_G$ is constructed by first breaking up $\partial\mathcal{T}_G$ into $[a]$, $[b]$, $[c]$, then breaking up $[a]$ into $a[b]$ and $a[c]$, and so on. Notice that if we delete the vertex 1 and its adjacent edges from $\mathcal{T}_G$, we get three infinite trees, and $\partial\mathcal{T}_G$ is partitioned into the three sets, and then deleting the base vertices a, b, c and their adjacent edges breaks up $[a]$, $[b]$, $[c]$ into two sets each, and so on.

The corresponding tree of subsets of $\overline{\mathsf{R}}$ breaks up $\overline{\mathsf{R}}$ into $R_a = [-\infty, 0]$, $R_b = [0,1]$, $R_c = [1, +\infty]$, which converts each of the three points 0, 1, ∞ of $\overline{\mathsf{R}}$ into two end-points. These intervals then break up by converting interior points into two end-points, with each successive $[\frac{p}{q}, \frac{r}{s}]$ breaking at the point $\frac{p+r}{q+s}$. This describes how R_a breaks up into aR_b and aR_c, and so on.

The corresponding tree of subsets of $\overline{\mathsf{C}}$ breaks up $\overline{\mathsf{C}}$ into the regions R_a, R_b, R_c, described in this section. Here R_c breaks up into cR_a and cR_b, and so on, as in Figure 9.3. The four regions aR_b, aR_c, R_b, R_c, are permuted by the four-group generated by a and d, which are represented as rotations of the Riemann sphere, so rotations carry the tree of subsets of R_c to those of R_b, aR_b, and aR_c.

The region R_c has two distinguished points, 1 and ∞, and these appear as points where the boundary is relatively smooth in Figure 1.1A. Similarly, the region R_a has two distinguished points, 0 and ∞; and, R_b has two distinguished points, 0 and 1.

The G-equivariant map $\Phi : \overline{\mathsf{R}} \to \overline{\mathsf{C}}$ carries $[-\infty, 0]$ to R_a, $[0,1]$ to R_b, and $[1,\infty]$ to R_c, and respects the tree of subsets. Thus Φ carries each interval occurring in the tree of subsets of $\overline{\mathsf{R}}$ to a Jordan region occurring in the tree of subsets of $\overline{\mathsf{C}}$. Figure 9.3 shows successive break-ups of the region R_c, and we now see how regions get successively filled in by the Peano curve, which

FIGURE 9.3. Successive Jordan partitions of R_c.

FIGURE 9.4. Images of successive approximations of Φ on $[1, \infty]$.

enters a Jordan region at one of the distinguished points, fills in the region, and leaves by the other distinguished point.

We can use the distinguished points to construct approximations to Φ. Thus the first approximation carries 0 to 0, 1 to 1, and ∞ to ∞, and we can extend this to get the usual inclusion $\overline{\mathsf{R}} \to \overline{\mathsf{C}}$. For the next approximation, we consider the intermediate points $a(1) = -1$, $b(\infty) = \frac{1}{2}$ and $c(0) = 2$ in $\overline{\mathsf{R}}$, which Φ respectively sends to $a(1) = -1$, $b(\infty) = -\omega$, $c(0) = 1 + \omega$ in $\overline{\mathsf{C}}$. We now have six values specified, and can extend this to a map $\overline{\mathsf{R}} \to \overline{\mathsf{C}}$; this can be done naturally once the Riemannian metrics are chosen. The image will consist of geodesics on the Riemann sphere, which join up the distinguished points of the Jordan regions.

We can continue subdividing the intervals in $\overline{\mathsf{R}}$, and the Jordan regions in $\overline{\mathsf{C}}$, and obtain successive approximations of Φ, as in the upper half of Figure 9.4. Of course we are not considering the Jordan regions, only the distinguished points; that is, we are considering a recursively defined bijection between the G-orbit $\mathsf{Q} \cup \{\infty\}$ of ∞ in $\overline{\mathsf{R}}$, and the G-orbit $\mathsf{Q}(\omega) \cup \{\infty\}$ of ∞ in $\overline{\mathsf{C}}$. Thus the approximations can be calculated directly, as in the lower half of Figure 9.4, and in [Th2]. The pictures suggests that these approximations will all give *embeddings* of $\overline{\mathsf{R}}$ into $\overline{\mathsf{C}}$, but we have not been able to prove that this is the case.

10. The action of $PGL_2(\mathsf{Z}[\omega]) \rtimes C_2$ on $\mathbf{H}^3$

In this section we prove some results about the action of $PGL_2(\mathsf{Z}[\omega]) \rtimes C_2$ which will be used in Section 11 to prove Theorem A.

10.1 Definitions. Let x be a point of $\partial\mathbf{H}^3$. A *horosphere centered at* x is a connected two-dimensional subvariety of $\mathbf{H}^3$ perpendicular to all the geodesics which have x as an end point, and is maximal with these properties. A *horoball centered at* x is a convex set bounded by a horosphere centered at x.

Throughout this section, we shall view elements of $PGL_2(\mathsf{Z}[\omega])$ as elements of $PSL_2(\mathsf{Z}[\omega]) \cup PSL_2(\mathsf{Z}[\omega]) \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$, as in Lemma 6.2.

10.2 Lemma. *Let p be an element of $\mathsf{Q}(\omega) \cup \{\infty\}$, and H a horoball centered at p. For any sequence of elements γ_n of $PGL_2(\mathsf{Z}[\omega])$, if the $\gamma_n(p)$ are all distinct, and form a sequence that converges to some $q \in \partial\mathbf{H}^3$, then, for every $x \in \mathbf{H}^3$, the sequence of $\gamma_n(x)$ tends to q, and the convergence is uniform on the horoball H.*

Proof. By applying an element of $PGL_2(\mathsf{Z}[\omega])$, we may assume that $p = \infty$ and $q \neq \infty$.

FIGURE 10.1. The ideal tetrahedron Δ_0.

For each n, let $\gamma_n = \pm \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Then $\gamma_n(p) = \frac{a_n}{c_n}$. Since the squence of $\gamma_n(p)$ has no repetitions, the sequence of c_n converges to infinity.

Let $x = z + t\,j$ be any point of $\mathbf{H}^3$. Since $a_n d_n - b_n c_n = 1$,

$$\gamma_n(x) = \gamma_n(z + t\,j) = \frac{a_n}{c_n} + \frac{-(\overline{c_n z + d_n})}{c_n(|c_n z + d_n|^2 + |c_n|^2 t^2)} + \frac{t\,j}{|c_n z + d_n|^2 + |c_n|^2 t^2}.$$

Hence

$$|\gamma_n(x) - \gamma_n(p)|^2 = |\gamma_n(x) - \frac{a_n}{c_n}|^2 = \frac{|c_n z + d_n|^2 + |c_n|^2 t^2}{|c_n|^2(|c_n z + d_n|^2 + |c_n|^2 t^2)^2} \leq \frac{1}{|c_n|^4 t^2}.$$

Since the sequence of c_n tends to infinity, we see that the sequence of $\gamma_n(x) - \gamma_n(p)$ tends to zero, and so the sequence of $\gamma_n(x)$ converges to q. Moreover, since a horoball H centered at ∞ is the set $\{z + tj \in \mathbf{H}^3 \mid t \geq t_0\}$, for some fixed $t_0 > 0$, the convergence is uniform on H.

10.3. Let Δ_0 be the ideal tetrahedron of $\mathbf{H}^3$ having ideal vertices 0, 1, $-\omega$, and ∞, as in Figure 10.1.

Let Γ denote the group of isometries of $\mathbf{H}^3$ generated by the reflections in the faces of Δ_0. It follows from Poincaré's Theorem that the Γ-orbit of Δ_0 gives a tessellation of $\mathbf{H}^3$.

We will see, in Section 11, that Δ_0 is a fundamental domain of the Gieseking manifold M, and that the $\pi_1(M)$-orbit of Δ_0 gives the same tessellation.

10.4 Proposition. *The above tessellation is preserved by $PGL_2(\mathbb{Z}[\omega]) \rtimes C_2$, where C_2 is the order two cyclic group generated by complex conjugation on $\overline{\mathbb{C}}$.*

Proof. Recall that Γ is the group generated by the reflections in the faces of Δ_0. Let Σ_4 be the group of isometries of $\mathbf{H}^3$ which act as the symmetric group on the set of vertices of Δ_0. We shall prove

$$PGL_2(\mathbb{Z}[\omega]) \rtimes C_2 = \Gamma \rtimes \Sigma_4.$$

It is easily checked that the given generators of $\Gamma \rtimes \Sigma_4$ lie in $PGL_2(\mathbb{Z}[\omega]) \rtimes C_2$, and it remains to prove that we get the whole group.

Consider any $\gamma \in PGL_2(\mathbb{Z}[\omega]) \rtimes C_2$.

Then $\gamma(\infty) \in \mathbb{Q}(\omega) \cup \{\infty\}$, and we claim that there exists $\mu \in \Gamma \rtimes \Sigma_4$ such that $\mu(\gamma(\infty)) = \infty$. Let Γ_∞ be the subgroup of Γ generated by the reflections in the three faces of Δ_0 containing ∞. Then Γ_∞ acts on $\mathbb{C}$ as Euclidean isometries, and the Euclidean triangle with vertices 0, 1, and $-\omega$, is a fundamental domain. In particular, if $\gamma(\infty) \neq \infty$, then there exists a Euclidean translation $\mu_0 \in \Gamma_\infty$ such that $|\mu_0(\gamma(\infty))| < 1$. It is not difficult to see that the Möbius transformation d, which acts by $z \mapsto 1/z$, lies in $\Gamma \rtimes \Sigma_4$. Hence $(d\mu_0\gamma)(\infty)$ either is ∞ or has modulus bigger than one. In the former case we are done, and in the latter case we iterate the process, which must terminate because the moduli of the denominators are strictly decreasing; this is just the Euclidean algorithm in $\mathbb{Z}[\omega]$. Hence there exists $\mu \in \Gamma \rtimes \Sigma_4$ such that $\mu\gamma(\infty) = \infty$.

Since $\mu\gamma \in PGL_2(\mathbb{Z}[\omega]) \rtimes C_2$, we see $\mu\gamma(0) \in \mathbb{Z}[\omega]$. By composing with a Euclidean translation in Γ_∞, we may assume that $\mu\gamma(0) = (0)$. So $\mu\gamma$ is a diagonal $\pm$-matrix composed with the identity or complex conjugation. Since the diagonal entries are units of $\mathbb{Z}[\omega]$, we have a finite number of possibilities for $\mu\gamma$, and it is easily seen that $\mu\gamma \in \Gamma \rtimes \Sigma_4$. Hence $\gamma \in \Gamma \rtimes \Sigma_4$, and this finishes the proof of the proposition.

10.5 Lemma. *If a sequence of elements γ_n of $PGL_2(\mathbb{Z}[\omega]) \rtimes C_2$ has the property that, for some $x_0 \in \mathbf{H}^3$, the sequence $\gamma_n(x_0)$ converges to a limit $x_\infty \in \partial\mathbf{H}^3$, then, for each ideal tetrahedron Δ of the above tessellation, the sequence $\gamma_n(x)$ tends to x_∞ uniformly for all $x \in \Delta$.*

Proof. It is well-known that $\gamma_n(x)$ tends to x_∞ for every $x \in \mathbf{H}^3$, and all that has to be proved is the uniformity of the convergence on Δ. By composing with isometries in $PGL_2(\mathbb{Z}[\omega]) \rtimes C_2$, we may assume that $\Delta = \Delta_0$, and that $x_\infty \neq \infty$. Moreover, up to taking a subsequence and composing with some isometry of the symmetry group of Δ_0, we may assume that each γ_n is orientation-preserving. For each n, let $\gamma_n = \pm \left(\begin{smallmatrix} a_n & b_n \\ c_n & d_n \end{smallmatrix}\right)$, so

$$\gamma_n(j) = \frac{b_n \overline{d}_n + a_n \overline{c}_n + j}{|c_n|^2 + |d_n|^2}.$$

Since $\gamma_n(j)$ tends to $x_\infty \in \mathsf{C}$, we see that the sequence of $|c_n|^2 + |d_n|^2$ tends to infinity. Moreover, since the entries all lie in either $\mathsf{Z}[\omega]$ or $\mathsf{Z}[\omega]i$, and satisfy $a_n d_n - b_n c_n = 1$, the product $c_n d_n$ is nonzero, for n sufficiently large, and in fact the sequence of $c_n d_n$ tends to infinity too. Since

$$|\gamma_n(0) - \gamma_n(\infty)| = |\frac{a_n}{c_n} - \frac{b_n}{d_n}| = \frac{1}{|c_n d_n|},$$

we see that the sequence of $|\gamma_n(0) - \gamma_n(\infty)|$ tends to zero. It follows from similar computations that the sequence of distances between the images of any other pair of vertices of Δ_0 tends to zero also. In particular, each $\gamma_n(\Delta_0)$ is the convex hull of a finite set contained in a disc, and the sequence of diameters of these discs tends to zero. The uniformity of the convergence on Δ_0 now follows from Lemma 6.6.

11. The Gieseking Manifold

11.1. We use the description given in [Mag].

Let Δ_0 be the ideal tetrahedron of $\mathbf{H}^3$ having vertices 0, 1, ∞, and $-\omega$ in $\overline{\mathsf{C}}$. Let $V = f^{-1}$, and $U = fba = bcf$, in $\widehat{G}$, so these act on $\overline{\mathsf{C}}$ by

$$\begin{aligned} & U(z) = \overline{\omega z + 1} = -(1+\omega)\overline{z} + 1 \\ \text{and} \quad & V(z) = (\overline{1 - \omega z})^{-1} = (1 + (1+\omega)\overline{z})^{-1}. \end{aligned}$$

As isometries of $\mathbf{H}^3$, U and V define identifications of the faces of Δ_0, since $U\colon(-\omega, 0, \infty) \mapsto (-\omega, 1, 0)$, $V\colon(1, 0, \infty) \mapsto (-\omega, 1, \infty)$.

Since the six edges of Δ_0 are all identified to each other, and all the dihedral angles are $\pi/3$, these identifications of the faces of Δ_0 define a cusped manifold M, called the *Gieseking manifold.*

11.2 Proposition. *The Gieseking manifold M is a fibre bundle over S^1, with fibre a punctured torus, and homological monodromy $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$. Moreover, the holonomy of the fibre is freely generated by ba and bc.*

Proof. By Poincaré's Theorem, the fundamental group of M has the presentation

$$\pi_1(M) = \langle U, V \mid VU = U^2V^2\rangle.$$

Let M' be the three-manifold fibered over S^1 with fibre a punctured torus and homological monodromy $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$. We want to show that M' is homeomorphic to M. The fundamental group of M' has presentation

$$\pi_1(M') = \langle r, s, t \mid trt^{-1} = s, tst^{-1} = sr\rangle,$$

which can be identified with $\pi_1(M)$ by identifying $r = UV$, $s = VU$, and $t = V$. Since both M and M' are the interiors of aspherical compact Haken manifolds, we conclude that they are homeomorphic, by Waldhausen's theorem.

The group of the fibre is the commutator subgroup of $\pi_1(M)$. Since $V = f^{-1}$, $U = fba = bcf$, it follows that the holonomy of the fibre is freely generated by ba and bc.

In [Ota] there is an explicit description of the fibration, due to Christine Lescop.

11.3. *Proof of Theorem A.* Let F be the fibre, and suppose that F is embedded in M. Let $\iota\colon \mathbf{H}^2 \to \mathbf{H}^3$ be the embedding induced by the inclusion $F \subset M$. We want to show that ι extends to a continuous $\pi_1(F)$-equivariant map $\Psi\colon \partial\mathbf{H}^2 \to \partial\mathbf{H}^3$.

We will discuss the general case in Lemma 11.7, but we begin by considering the case where the hyperbolic structure of F is the one induced by the action of G on $\mathbf{H}^2$ given in the Introduction; that is, the structure induced by the Farey tessellation. In this case, we want to show that ι extends to the map Φ given by Theorem C.

11.4 Lemma. *Consider the action of $\pi_1(F) \subset G$ on $\overline{\mathsf{R}}$ and on $\overline{\mathsf{C}}$. For any $\xi \in \mathsf{Q} \cup \{\infty\}$, and $\gamma \in \pi_1(F)$, γ fixes ξ if and only if γ fixes $\Phi(\xi)$.*

Proof. Let $\Gamma_\xi \le \pi_1(F)$, and $\Gamma_{\Phi(\xi)} \le \pi_1(F)$, denote the stabilizers of ξ, and $\Phi(\xi)$, respectively. By $\pi_1(F)$-equivariance, $\Gamma_\xi \le \Gamma_{\Phi(\xi)}$. Since both actions of $\pi_1(F)$ are discrete and faithful, these stabilizers are maximal abelian subgroups. Thus they are equal. (In fact, this was seen in an explicit form in the proof of Proposition 8.11.)

For any $x \in M$, let $\mathrm{inj}_M(x)$ denote the *injectivity radius* of x in M, that is, $\mathrm{inj}_M(x)$ is the maximal radius of an open ball centered at x, and embedded in M. The next result follows immediately from Margulis' Lemma; see, for example, [Th1, §§5.10, 5.11].

11.5 Lemma. (i) *For every* $\varepsilon > 0$, $\{x \in F \mid \mathrm{inj}_M(x) \geq \varepsilon\}$ *is compact.*
(ii) *There exist* $\varepsilon_M > 0$ *such that* $\{x \in M \mid \mathrm{inj}_M(x) \leq \varepsilon_M\}$ *lifts back to a family of disjoint horoballs in the universal cover* $\mathbf{H}^3$.

Let δ be the ideal triangle of $\mathbf{H}^2$ having vertices 0, 1, and ∞. In Section 3, we saw that δ is a fundamental domain for the G-action on $\mathbf{H}^2$. Let $D = \delta \cup a\delta$, the ideal quadrilateral with vertices 0, 1, ∞, and -1. By Lemma 11.5, we have a decomposition

$$D = D_0 \cup D_1 \cup D_2 \cup D_3 \cup D_4,$$

where D_n is connected for $n = 0, \dots, 4$, D_0 is compact, and, for each x in $\bigcup_{n=1}^{4} D_n$, $\mathrm{inj}_M(p\iota(x)) < \varepsilon_M$. In particular, for $n = 1, \dots, 4$, $\iota(D_n)$ is contained in a horoball.

11.6. Let $(x_n)_{n \geq 0}$ be a sequence in $\mathbf{H}^2$ converging to a limit $x_\infty \in \partial\mathbf{H}^2$. We claim that the sequence of $\iota(x_n)$ tends to $\Phi(x_\infty)$. Up to taking a subsequence, we distinguish two cases:
Case 1. Assume that, for each $n > 0$, $x_n \in \pi_1(F)D_0$. That is, for each $n > 0$, we have $w_n \in \pi_1(F)$ such that $x_n \in w_n D_0$. For any fixed $x_0 \in \mathbf{H}^2$, the sequence of $w_n x_0$ tends to x_∞, since D_0 is compact. Thus, by Theorem C, $w_n \iota(x_0)$ tends to $\Phi(x_0)$.

Since D_0 is compact, there exist $\gamma_1, \dots, \gamma_k \in \pi_1(M)$, such that $\iota(D_0)$ is contained in $\gamma_1\Delta_0 \cup \cdots \cup \gamma_k\Delta_0$. So, for each $n \in \mathsf{N}$,

$$\iota(x_n) \in w_n\gamma_1\Delta_0 \cup \cdots \cup w_n\gamma_k\Delta_0,$$

by $\pi_1(F)$-equivariance. By Lemma 10.5, $w_n\iota(D_0)$ converges to $\Phi(x_\infty)$. We conclude that the sequence of $\iota(x_n)$ tends to $\Phi(x_\infty)$, since, by Lemma 10.5, for each $x \in \mathbf{H}^3$, the sequence of $w_n x$ tends to $\Phi(x_\infty)$, and the convergence is uniform on $\gamma_1\Delta_0 \cup \cdots \cup \gamma_k\Delta_0$. This completes Case 1.
Case 2. Assume that, for each $n > 0$, $x_n \in \pi_1(F)D_1$. We may assume that the sequence of $\mathrm{inj}_M(p\iota(x_n))$ tends to zero, where $p\colon \mathbf{H}^3 \to M$ is the projection of the universal covering, because, otherwise we would be in Case 1, by Lemma 11.5 (i).

Let $p_1 \in \overline{\mathsf{R}} = \partial\mathbf{H}^2$ be an ideal point in the closure of D_1. By hypothesis, p_1 is one of the ideal vertices of D. Up to a taking a subsequence, we have two subcases.

Subcase 2.1. Assume there exists $\gamma_0 \in \pi_1(F)$ such that, for every $n > 0$, $x_n \in \Gamma\gamma_0 D_1$, where $\Gamma < \pi_1(F)$ is the stabilizer of $\gamma_0(p_1)$. By $\pi_1(F)$-equivariance, we may assume $\gamma_0 = 1$. The limit point x_∞ is p_1, because p_1 is the only ideal point in the closure of ΓD_1.

By Lemma 11.5, $\iota(D_1)$ is contained in a horoball. Since ΓD_1 is connected, the set $\iota(\Gamma D_1)$ is contained in a horoball of $\mathbf{H}^3$ whose center is fixed by Γ. By $\pi_1(F)$-equivariance and Lemma 11.4, the center of this horoball is $\Phi(p_1)$. Since the sequence of $\mathrm{inj}_M(p\iota(x_n))$ tends to zero, the sequence of $\iota(x_n)$ tends to the center, $\Phi(p_1)$, of the horoball.

Subcase 2.2. Assume that, for every $n > 0$, there exists $\gamma_n \in \pi_1(F)$ such that $x_n \in \gamma_n D_1$, and $\gamma_n(p_1) \neq \gamma_m(p_1)$ if $n \neq m$. Since p_1 is a vertex of the Farey tessellation, the sequence $\gamma_n(p_1)$ converges to x_∞. Hence the sequence of $\gamma_n \Phi(p_1)$ tends to $\Phi(x_\infty)$.

As in Subcase 2.1, it can be shown, by an equivariance argument, that $\iota(D_1)$ is contained in a horoball H centered at $\Phi(p_1)$. Moreover, if $m \neq n$, then $\gamma_n\Phi(p_1) \neq \gamma_m\Phi(p_1)$, by Lemma 11.4. So, by Lemma 10.2, for every $x \in H$, the sequence of $\gamma_n(x)$ tends uniformly to $\Phi(x_\infty)$. Hence, since $\iota(x_n)$ is contained in $\gamma_n H$ for every $n > 0$, the sequence $\iota(x_n)$ tends to $\Phi(x_\infty)$.

This completes Case 2, and the proof of Theorem A in the case where the fibre has the hyperbolic structure coming from the Farey tessellation.

The following result reduces the general case to the previous one.

11.7 Lemma. *For any two actions of the free group of rank two on $\mathbf{H}^2$ by isometries, such that the quotients are (finite volume, complete) punctured tori, the actions are conjugate by a homeomeorphism which extends to an equivariant homeomorphism of $\partial\mathbf{H}^2$.*

Proof. Let S_1 and S_2 be the two punctured tori obtained as quotients. By Teichmüller theory, there is a quasiconformal homeomeorphism $h\colon S_1 \to S_2$; see, for example, [Abi]. The lift $\tilde{h}\colon \mathbf{H}^2 \to \mathbf{H}^2$ is a quasiconformal homeomorphism which conjugates one action to the other. Since $\tilde{h}$ is quasiconformal, it extends to a homeomorphism of $\mathbf{H}^2 \cup \partial\mathbf{H}^2$.

This completes the proof of Theorem A.

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