UNIFORM GROWTH, ACTIONS ON TREES AND GL_2

Roger C. Alperin and Guennadi A. Noskov

1. Exponential Growth

Choose a finite generating set $S = \{s_1, \cdots, s_p\}$ for the group Γ; define the $S-length$ of an element as $\lambda_S(g) = min\{\ n \mid g = s_1 \cdots s_n, s_i \in S \cup S^{-1}\}$. The growth function $\beta_n(S, \Gamma) = |\{g \mid \lambda_S(g) \leq n\}|$ depends on the chosen generating set. A group has exponential growth if the growth rate, $\beta(S, \Gamma) = limit_{n\to\infty}\beta_n(S, \Gamma)^{\frac{1}{n}}$ is strictly greater than 1. In fact, for another finite generating set $T = \{t_1, \cdots, t_q\}$ for Γ, if both $max_j \lambda_S(t_j) \leq L$ and $max_i \lambda_T(s_i) \leq L$, then $\beta_n(S, \Gamma) \leq \beta_{Ln}(T, \Gamma)$ and also the symmetric inequality. It then follows that $\beta(S, \Gamma)^L \leq \beta(T, \Gamma)$ and $\beta(T, \Gamma)^L \leq \beta(S, \Gamma)$. Using these remarks, Milnor showed that exponential growth is independent of the generating set.

For a group with exponential growth we consider

$$\beta(\Gamma) = inf_S \beta(S, \Gamma).$$

If $\beta(\Gamma) > 1$ then Γ is said to have *uniform exponential growth*.

Gromov has asked if there is a group of exponential growth which is not of uniform exponential growth. Indications are that such a group will be hard to find. Recently, [1, 9], it has been shown that all solvable groups which have exponential growth are of uniform exponential growth.

1991 *Mathematics Subject Classification.* 20E36,20F32.

Key words and phrases. exponential growth, trees, valuations, uniform exponential growth.

The authors thank the Mathematics Department at Bielefeld University, Germany, for their hospitality and support of this research project. G. Noskov supported by GIF-grant G-454-213.06/95, Gr-627/9.

2. Generalities

The following uses the fact that given a generating set S for Γ the set of elements of the subgroup $\mathcal{H}$, a subgroup of index d, which are words in S of length at most $2d-1$ give a generating set for $\mathcal{H}$.

Proposition 2.1 ([10])**.** *If a group Γ has a subgroup $\mathcal{H}$ of finite index d, then $\beta(\Gamma) \geq \beta(\mathcal{H})^{\frac{1}{2d-1}}$.*

The next proposition is elementary, but also a very useful result.

Proposition 2.2. *If a group Γ has homomorphic image which is of uniform exponential growth then Γ has uniform exponential growth.*

A group is called *large* if it has a homomorphism onto a free non-abelian group. It is important to realize that a free group of rank n has uniform exponential growth of rate $\beta = 2n-1$.

Corollary 2.3. *If the group Γ is virtually large then Γ has uniform exponential growth.*

There has been some interesting recent work on linear groups which are large, [7], [8], in a sense building on some ideas from [11].

We say that a group Γ has the *UF-property* (uniformly contains a free nonabelian semigroup) if there is a constant $n_\Gamma \geq 1$ such that for every generating set S of Γ there exist two elements (depending on S) in Γ of word length $\leq n_\Gamma$ and freely generating a free semigroup of rank 2.

Proposition 2.4. *If a group Γ has the UF-property then Γ is of uniform exponential growth.*

Proof. Let S be an arbitrary finite generating set for Γ and let $S_0 = \{g, h\}$ be the pair of words of $S-\mathit{length}$ less than or equal to n_Γ and freely generating a free semigroup Γ_0 of rank 2. We have

$$\beta(S,\Gamma)^{n_\Gamma} = \beta(S \cup S_0, \Gamma) \geq \beta(S_0, \Gamma_0) \geq 2$$

and the proof is complete. ■

3. Action on Trees

We use a variant on the usual ping-pong lemma to obtain free semigroups.

Lemma 3.1 ([4])**.** *Let Γ_0 be a group of isometries of a tree X with the generating set $\{g_1,\ g_2\}$ where g_1 g_2 are hyperbolic isometries with distinct axes A_1, A_2. Then one of the four pairs $\{g_1^{\pm 1}, g_2^{\pm 1}\}$ freely generate a free semigroup of rank two.*

Proof. If A_1, A_2 are disjoint then let $[a_1, a_2]$ be the shortest geodesic joining them. Let A_1^+, A_2^+ be the rays, starting at a_1, a_2, such that $g_1,\ g_2$ translate towards the ends of these rays. Let $R_1^+,\ R_2^+$ be the rays $A_1^+,\ A_2^+$ with the first unit length segments removed. Let $p_1,\ p_2$ be the geodesic projection maps of X onto $R_1^+,\ R_2^+$ respectively. Set $X_i = p_i^{-1}(R_i^+),\ i = 1, 2$. Clearly $X_1 \cap X_2 = \emptyset$ and $g_i^n(X_1 \cup X_2) \subset X_i,\ n \geq 1, i = 1, 2$. We assert now that $g_1,\ g_2$ freely generate a free semigroup of rank 2. Indeed let $w_1,\ w_2$ be (positive) words in $g_1,\ g_2$ and suppose $w_1 = w_2$ in Γ_0. If the words start with the same letter, the final segments are also equal in Γ_0; hence by induction we get that $w_1 = w_2$ as formal words. Thus, we may assume that the words $w_1,\ w_2$ start with different letters, say $g_1,\ g_2$ respectively. Then the image of $X_1 \cup X_2$ under $w_1,\ w_2$ belongs to $X_1,\ X_2$ respectively; hence a contradiction. ■

Theorem 3.2. *Suppose that Γ is a finitely generated subgroup of $\mathrm{GL}_2(K)$ over the field K.*

(char $\neq 0$) *If K has nonzero characteristic and Γ has exponential growth then Γ satisfies the UF-property and consequently has uniform exponential growth.*

(char $= 0$) *If K has characteristic zero and Γ has exponential growth then either Γ has uniform exponential growth or Γ is conjugate to a subgroup of $\mathrm{GL}_2(\mathcal{O})$, for $\mathcal{O}$ a ring of integers in an algebraic number field.*

Proof. Since Γ is finitely generated we may assume that K is finitely generated. For any discrete rank one valuation v of K let X_v be the corresponding Bruhat-Tits tree. We consider the action of Γ (or a subgroup of index 2, acting without inversions) on each of the Bruhat-Tits trees X_v for each of these valuations and split the proof of the theorem into cases.

(1) For any valuation v, Γ has a fixed point on X_v . In this case, the ring A generated by the traces of elements of Γ lies in every valuation ring A_v of K. In non-zero characteristic this ring is finite, [11] I.6.2, generated over the prime field by roots of unity, so in fact there are only finitely many traces and it follows that Γ is finite, [12] 1.20, (hence of eventually constant growth), if the group acts irreducibly, or it is solvable if the group acts reducibly and hence also of uniform exponential growth, [9]. In characteristic zero, the ring of traces is the ring of algebraic integers $\mathcal{O}$ in the algebraic closure of $\mathbb{Q}$ in K; as in [3], one can find a suitable module so that the group is conjugated into $\mathrm{GL}_2(\overline{\mathcal{O}})$, for a somewhat larger ring of integers $\overline{\mathcal{O}}$.

(2) There is a valuation such that Γ has an invariant line on X_v. Then by [11] II.1.3, Γ is contained in a Cartan subgroup, which is an extension of diagonal subgroup by cyclic of order 2. Hence Γ is virtually abelian and has polynomial growth.

(3) For some valuation v there is neither a fixed point nor an invariant line on the Bruhat-Tits tree X_v. We prove that the image of Γ (that is modulo scalar matrices) satisfies the UF-property with a constant $n_\Gamma = 6$ and hence so does the original group.

    Let S be a finite generating set of Γ. The subcases are as follows.

    (a) S contains a hyperbolic isometry, say g and any $s \in S$ leaves the axis A_g invariant. Then we are in case 2.

    (b) S contains a hyperbolic isometry, say g and there is $s \in S$ such that $sA_g \neq A_g$. The isometry $h = sgs^{-1}$ is hyperbolic with the axis $sA_g \neq A_g$, hence by Lemma 3.1 one of the four pairs $\{g^{\pm 1}, h^{\pm 1}\}$ freely generate a free semigroup of rank 2 and the length of $g^{\pm 1}$ and $h^{\pm 1}$ is at most 3.

    (c) S does not contain a hyperbolic isometry, that is all isometries are elliptic, thus any $s \in S$ has a fixed vertex in X_v. Then there exists $g \in S \cup S^2$ which is hyperbolic, [11], I.6.4. Again we may assume there is $s \in S \cup S^2$ such that $sA_g \neq A_g$. The argument used for the previous case shows now that one of the four pairs $\{g^{\pm 1}, sg^{\pm 1}s^{-1}\}$ freely generate a free semigroup of rank 2 and the length of $g^{\pm 1}$, respectively, $sg^{\pm 1}s^{-1}$ is at most 6 and the proof is complete. ■

4. Incidentals

In the characteristic zero case, we have shown that the finitely generated group Γ of exponential growth is of uniform exponential growth or it is conjugate to a subgroup of $\mathrm{GL}_2(\mathcal{O})$, for $\mathcal{O}$ a ring of integers in an algebraic number field. This however is not a dichotomy. Certainly, there are many cases of discrete subgroups of $\mathrm{GL}_2(\mathbb{C})$ which are of uniform exponential growth; these arise as the fundamental group of a hyperbolic manifold [6], [10].

For further analysis we consider the case when every element of Γ is elliptic (considered as a group of Mobius transformations). By dint of a theorem of Lyndon-Ullman [5] the group is conjugate (via stereographic projection) to a group of rotations of the sphere. However, for a linear group which has compact closure, every eigenvalue of every element must be on the unit circle. If we are also in the case where Γ is a subgroup of $\mathrm{GL}_2(\mathcal{O})$, then under every embedding $\sigma : \mathcal{O} \to \mathbb{C}$ the above property of eigenvalues holds for $\sigma(\Gamma)$, since ellipticity is preserved. Hence every eigenvalue is a root of unity by Dirchlet's Theorem. By passing to a subgroup of finite index (using compactness) we can assume the determinant of the matrices in this group are all unity. Thus the trace of the matrices are in fact a sum of a root of unity and its complex conjugate. But now there are only finitely many traces since the fraction field of $\mathcal{O}$ has finite dimension over $\mathbb{Q}$ and only finitely many roots of unity can belong to a number field of given degree. It now follows [2], that Γ is finite since we can pass to a subgroup of finite index missing these finitely many possible traces $\neq 2$. But this subgroup of finite index has elements with all eigenvalues which are 1; thus every element is unipotent and so must trivial by compactness.

We summarize these remarks in the following statement.

Corollary 4.1. *Suppose that Γ is a finitely generated subgroup of $\mathrm{GL}_2(\mathbb{C})$ of exponential growth and not of uniform exponential growth then Γ can not consist entirely of elliptic elements.*

5. Related Open Problems

Question 1. *Prove uniform exponential growth for linear groups of exponential growth. For example show uniform exponential growth for $SL_2(\mathcal{O})$ where $\mathcal{O}$ is the ring of integers of a real quadratic number field.*

Question 2. *Prove that uniform exponential growth is a quasi-isometry invariant.*

Question 3. *Classify or otherwise characterize the groups which have the property that every generating set has two elements which generate a free semigroup. (if torsion-free are they of finite cohomological dimension?)*

References

[1] R. C. Alperin, Uniform growth of polycyclic groups, preprint 2000
[2] R. C. Alperin, Two dimensional representations of groups with property FA, Proc. Amer. Math. Soc. 108, No. 1, Jan. 1990, 283-4
[3] H. Bass, Finitely generated subgroups of GL_2, **The Smith Conjecture** edited by H. Bass and J. W. Morgan, Academic Press, 1984
[4] M. Bucher and P. de la Harpe, Free products with amalgamation and HNN extensions which are of uniformly exponential growth, preprint 2000
[5] R. C. Lyndon and J. L. Ullman, Groups of elliptic linear fractional transformations, Proc. Amer. Math. Soc. 18, 1967, 1119-1124
[6] F. Grunewald and G.Noskov, Largeness of certain hyperbolic lattices, preprint 2000
[7] A. Lubotzky, Free quotients and the first betti number of some hyperbolic manifolds, Transformation Groups, 1,1996,71-82
[8] G. A. Margulis and E. B. Vinberg, Some linear groups virtually have a free quotient, J. Lie Theory, 10, 2000, 171-180
[9] D. Osin, The entropy of solvable groups, preprint 2000
[10] P. B. Shalen, P. Wagreich, Growth rates, $\mathbb{Z}_p$-homology, and volumes of hyperbolic 3-manifolds, Trans. Am. Math. Soc., 331, No. 2, 1992, 895-917
[11] J-P. Serre, **Trees**, Springer-Verlag, 1980.
[12] B. A. F. Wehrfritz, **Infinite Linear Groups**, Springer-Verlag, 1973.

Department of Mathematics and Computer Science
San Jose State University, San Jose, CA 95192, USA
E-mail: alperin@mathcs.sjsu.edu

Institute of Mathematics, Russian Academy of Sciences
Pevtsova 13, Omsk 644099, RUSSIA
E-mail: noskov@private.omsk.su