Nonvanishing of algebraic entropy for geometrically finite groups of isometries of Hadamard manifolds *

Roger C. Alperin
San Jose State University, San Jose, CA, USA

Gennady A. Noskov
Bielefeld University, GERMANY and
Institute of Mathematics, Omsk, RUSSIA

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Abstract

We prove that any nonelementary geometrically finite group of isometries of a pinched Hadamard manifold has nonzero algebraic entropy in the sense of M. Gromov. In other words it has a uniform exponential growth,

1 Introduction

Given a group Γ generated by a finite set S we denote by $B_S(r)$ the the ball of radius k in the Cayley graph of Γ relative to S. The exponential growth rate or $\omega(\Gamma, S) = \lim_{k\to\infty} \sqrt[k]{|B_S(k)|}$ is well defined (by submultiplicativity). We set

$$\omega(\Gamma) = \inf\{\omega(\Gamma, S) | S \text{ is a finite generating set for } \Gamma\}.$$

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The group Γ is of exponential growth if $\omega(\Gamma, S) > 1$ for some (and hence for every) generating set S and of uniform exponential growth if $\omega(\Gamma) > 1$. (M. Gromov calls $\ln \omega$ the entropy of Γ - the reason is that if Γ is the fundamental group of a compact Riemannian manifold of unit diameter, then $\ln \omega(\Gamma, S)$ is a lower bound for the topological entropy of the geodesic flow of the manifold [Man81]). This notion should not be confused with the one given in [Ave72], where the notion of entropy is defined for the random walk on the group associated to the probability measure on the set of generators.

Clearly uniform exponential growth implies exponential growth. "On first sight, however, there seems to be no reason for the converse to be true" [Gro99]. Recently and rather spectacularly, J. S. Wilson has given example of exponential growth group which is not uniform [Wil02]; it is a group of automorphisms of a rooted tree.

The most common way to prove uniformity of exponential growth is to prove the UFS-property. We say that Γ satisfies UFS-property (=uniformly contains free nonabelian semigroup) if there is a constant $n_\Gamma \geq 1$ such that for every generating set A of Γ there exist two elements in Γ of word length $\leq n_\Gamma$, freely generating a free semigroup of rank 2. The UFG-property, obtained from the previous one by changing free semigroup to free group is stronger.

Our main result is the following.

Theorem 1.1 *Any nonelementary geometrically finite group of isometries of a pinched Hadamard manifold satisfies the UFG-property and hence has nonzero entropy.*

We mention a few ideas about the proof of our main result given here. If Γ is cocompact then it is word hyperbolic and the theorem follows from Koubi's result [Kou98] (which was known before due to M. Gromov [Gro87] and T. Delzant [Del96] in the torsionfree case.) In the main case of a noncocompact lattice we make use the geometry of neutered space associated to the group.

For a survey on this subject see the book of Gromov and the articles [GdlH97],[dlH02]. We mention now some related results. A group is called large if it has a subgroup of finite index which has a homomorphism onto a free non-abelian group. For example, in [GG00], largeness is proven for the noncocompact lattices in 3-dimensional hyperbolic space.

Also in our article [AN02], partial results were obtained for subgroups of SL_2 in characteristic zero, but more complete results for non-zero characteristic.

Theorem 1.2 *If a finitely generated subgroup* Γ *of* $\mathrm{GL}_2(K), K$ *a field of nonzero characteristic has an exponential growth then it satisfies the UFG-property and consequently has uniform exponential growth.*

A. Eskin, S. Mozes and H. Oh have recently given the 'uniform' proof of Tits' theorem in their beautiful result.

Theorem 1.3 *[EMO01] Any finitely generated non-virtually solvable linear group over a field of characteristic zero has uniform exponential growth.*

2 Miscellany about δ-hyperbolic spaces

Let X be a metric space. We denote by $|x-y|$ the distance between the points x and y in X. If X is equipped with a basepoint x_0 then we shall use the notation $|x| = |x - x_0|, x \in X$. One of definitions of a hyperbolic space X is given by means of the Gromov product relative to a base point x_0

$$(x \cdot y) = (x \cdot y)_{x_0} = \frac{1}{2}(|x| + |y| - |x-y|), x, y \in X \tag{1}$$

as follows: A metric space X is called hyperbolic [Gro87] if there exists a constant $\delta \geq 0$ such that for every triple $x, y, z \in X$ and for every choice of a basepoint the following holds

$$(x \cdot y) \geq \min((x \cdot z), (y \cdot z)) - \delta. \tag{2}$$

A tripod T is a union of three segments in $\mathbb{R}^2$, which have only the origin in common. Every geodesic triangle $\Delta = xyz$ in a geodesic metric space X can be mapped onto a tripod so that the restriction of the map to each side of Δ is an isometry. We will call a map with these properties a tripod map. The tripod map always exists and is essentially unique. The three points x', y', z' on the sides opposite to x, y, z respectively, whose image by the tripod map is equal to 0 describe an inscribed triangle and are called the internal vertices of Δ. A geodesic triangle Δ is called δ-thin for $\delta \geq 0$, if the fibers of the tripod map $f : \Delta \to T$ are of diameter at most δ.

Lemma 2.1 *Any geodesic triangle in a* δ*-hyperbolic space* H *is* 4δ*-thin.*

Proof. See [ABC$^+$91] or [CDP90]. □

Let xyz be a geodesic δ-thin triangle with the inscribed triangle $x'y'z'$. Let $\sigma_y(t), \sigma_z(t)$ be the arc length parameterizations of the segments $[x,y],[x,z]$ respectively such that at the moment $t=0$ they are located in x. We extend the parameterization for all $t \geq 0$ making the paths stop when they reach the vertices. Then by definition of thinness and lemma 2.1 the points $\sigma_y(t), \sigma_z(t)$ are distance at most 4δ apart until the moment when they reach the vertices y', z' of the inscribed triangle, that is

$$|\sigma_y(t) - \sigma_z(t)| \leq 4\delta, 0 \leq t \leq T = \frac{1}{2}(|x-y| + |x-z| - |y-z|) = (y \cdot z)_x. \quad (3)$$

We say that $\sigma_y(t), \sigma_z(t)$ 4δ-fellow travel each other on the segment $[0,T]$. More generally we have the following

Definition 2.2 *The paths $\sigma(t), \tau(t)$ ε-*fellow travel *each other on the segment $[t_0, t_1]$ if $|\sigma(t) - \tau(t)| \leq \varepsilon,\ t_0 \leq t \leq t_1$. The function $\ell(\sigma(t), \tau(t), \varepsilon), \varepsilon > 0$ is the supremum of the lengths of all time intervals on which $\sigma(t), \tau(t)$ ε-fellow travel.*

Lemma 2.3 (Fellow traveller property) *Let xyz be a geodesic triangle in a $\delta-$hyperbolic space X and $|y-z| = c > 0$. Let $\sigma_y(t), \sigma_z(t)$ be the arc length parameterizations of the the segments $[x,y]$ and $[x,z]$. Then*

1) $|\sigma_y(t) - \sigma_z(t)| \leq c + 8\delta$ for all $t \geq 0$.

2) For the reverse parameterizations $\sigma_y^-(t), \sigma_z^-(t)$ which start from y, z respectively, we have $|\sigma_y^-(t) - \sigma_z^-(t)| \leq c + 8\delta, t \geq 0$.

3) Suppose that γ, γ' are geodesics with beginning and ending distance at most $c > 0$. We have $|\gamma(t) - \gamma'(t)| \leq 16\delta + 2c$ for all $t \geq 0$.

Proof. 1) Consider the inscribed triangle for xyz. If both $\sigma_y(t)$ and $\sigma_z(t)$ do not reach the vertices of inscribed triangle then $|\sigma_y(t) - \sigma_z(t)| \leq 4\delta$. If they do reach then there are points y', z' on the side y, z distance at most 4δ from $\sigma_y(t), \sigma_z(t)$ respectively hence $|\sigma_y(t) - \sigma_z(t)| \leq c + 8\delta$.

2) If both $\sigma_y^-(t)$ and $\sigma_z^-(t)$ do not reach the vertices of inscribed triangle then arguing as in the previous case we obtain that $|\sigma_y^-(t) - \sigma_z^-(t)| \leq c + 8\delta$. In contrast to the previous case it is possible, that $\sigma_y^-(t)$ has reached the vertex of inscribed triangle but $\sigma_z^-(t)$ has not. If this happens, then let z' be a point on $[z,x]$ such that $|\sigma_y^-(t) - x| = |z' - x|$. We have $|\sigma_z^-(t) - z'| = ||z-x| -$

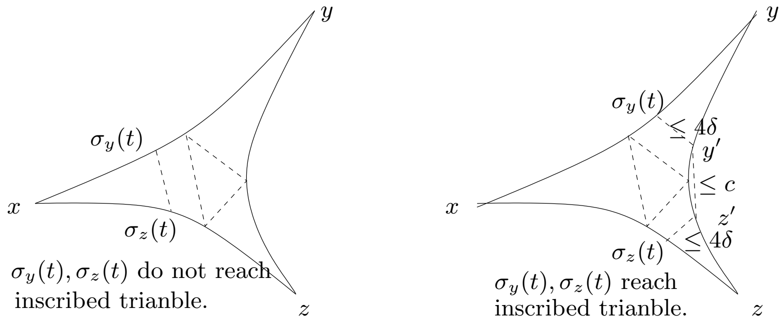

Figure 1: Fellow traveller property. Case 1).

$|y - x|| \leq c$, (see Figure 2). We conclude that $|\sigma_y^-(t) - \sigma_z^-(t)| \leq c + 4\delta, t \geq 0$. Finally if both $\sigma_y^-(t)$ and $\sigma_z^-(t)$ reach the vertices of inscribed triangle then again choosing suitable z' (or y') as above we get $|\sigma_y'(t) - \sigma_z'(t)| \leq c + 4\delta$.

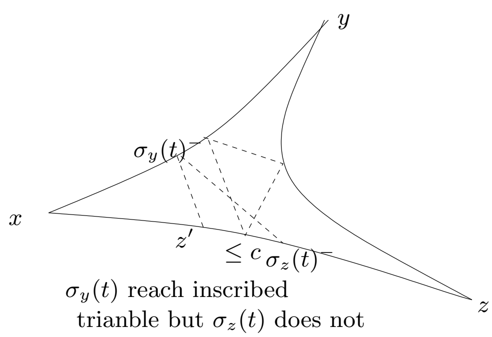

Figure 2: Fellow traveller property. Case 2)

3) Easily follows applying 1) and 2) to the triangles $\gamma(0)\gamma(\infty)\gamma'(\infty)$ and $\gamma'(0)\gamma(\infty)\gamma'(\infty)$ respectively. □

We say that the geodesic triangle xyz has an obtuse y-angle if either y is the nearest to x point on the side $[y, z]$ or y is the nearest to z point on the side $[y, x]$.

Lemma 2.4 (Obtuse-angled triangle lies in $8\delta-$neighbourhood of the side) *Let xyz be a geodesic triangle with obtuse y-angle, then $|x - z| \geq |x - y| + |y - z| - 16\delta$. Furthermore, $(x \cdot z)_y \leq 8\delta$.*

Proof. Use the inscribed triangle. Say y is nearest to x on the side $[y, z]$. Since $|z' - x'| \leq 4\delta$ we conclude that $|z' - y| \leq 4\delta$ hence $|y - y'| \leq 8\delta$.

Thus $|x - z| = |x - y'| + |y' - z| \geq |x - y| - |y - y'| + |y - z| - |y - y'| \geq |x - y| + |y - z| - 16\delta$. Finally, applying the last inequality we obtain $(x \cdot z)_y = \frac{1}{2}(|x - y| + |z - y| - |z - x|) \leq 8\delta$.

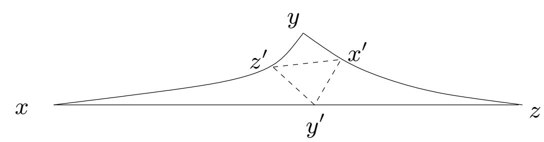

Figure 3: Obtuse triangle inequality.

□

The next lemmas help us figure out the structure of geodesic polygons; they are proven in greater generality in [Ol′91], see Lemmas 21 and 22.

Lemma 2.5 *1) Suppose that a geodesic 4-gon $[x_1, x_2, x_3, x_4]$ satisfies the following conditions: $|x_{i+1} - x_i| > 180\delta$ for $i = 1, 2, 3$ and $(x_1 \cdot x_3)_{x_2}, (x_2 \cdot x_4)_{x_3} \leq 14\delta$ for. Then the polygonal line $p = x_1x_2x_3x_4$ is contained in the closed 28δ-neighborhood of the side $[x_1, x_4]$. In particular, $|x_1 - x_4| > \sum_1^3 |x_{i+1} - x_i| - 168\delta$.*

2) Suppose that a geodesic 5-gon $[x_1, x_2, x_3, x_4, x_5]$ satisfies the following conditions: $|x_{i+1} - x_i| > 180\delta$ for $i = 1, 2, 3, 4$ and $(x_{i+2} \cdot x_i)_{x_{i+1}} \leq 14\delta$ for $i = 1, 2, 3$. Then the polygonal line $p = x_1x_2x_3x_4x_5$ is contained in the closed 28δ-neighborhood of the side $[x_5, x_1]$. In particular, $|x_1 - x_5| > \sum_1^4 |x_{i+1} - x_i| - 168\delta$.

We have the following variant of the above lemma for "small" sides $[x_2, x_3], [x_4, x_5]$.

Lemma 2.6 *Suppose that a geodesic 5-gon $[x_1, x_2, x_3, x_4, x_5]$ satisfies the conditions $|x_2 - x_3|, |x_4 - x_5| \leq 180\delta$. Then*

$$|x_1 - x_5| > |x_1 - x_2| + |x_3 - x_4| - 360\delta - 2\ell_0,$$

where $\ell_0 = \ell(\gamma, \gamma', 380\delta)$ - the time of 380δ-fellow travel of geodesics $\gamma = [x_2, x_1], \gamma' = [x_3, x_4]$.

Proof. First note that $\ell_0 \geq (x_1 \cdot x_5)_{x_2}$. Indeed, the geodesics along the segments $[x_2, x_1], [x_2, x_5]$ $4\delta-$fellow travel each other until the moment $t = (x_1 \cdot x_5)_{x_2}$ by the thinness property, see inequality (3). Geodesics

$[x_2, x_5], [x_3, x_4]$ 376δ-fellow travel each other all the time by the assertion 3) of Lemma 2.3. Thus the segments $[x_2, x_1], [x_3, x_4]$ 380δ-fellow travel each other until the moment $t = (x_1 \cdot x_5)_{x_2}$, hence $\ell_0 \geq (x_1 \cdot x_5)_{x_2}$. Finally, $|x_1 - x_5| = |x_1 - x_2| + |x_2 - x_5| - 2(x_1 \cdot x_5)_{x_2} \geq |x_1 - x_2| + |x_3 - x_4| - 360\delta - 2(x_1 \cdot x_5)_{x_2} > |x_1 - x_2| + |x_3 - x_4| - 360\delta - 2\ell_0$. □

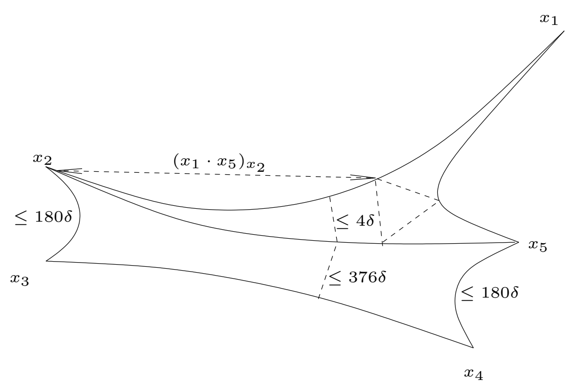

Figure 4: Lemma 2.6: Divergence for 5-gons with two small sides.

3 Uniformly hyperbolic isometries in groups acting on $\delta-$hyperbolic spaces

An isometry g of a $\delta-$hyperbolic space X is hyperbolic if there is a point $x \in X$ such that the map $n \mapsto g^n x$ from $\mathbb{Z}$ to X is a quasi-isometric embedding, [CDP90]. This means that $|g^n(x) - x| \geq cn$ for a suitable positive constant c and every natural n. The axis of g is a bi-infinite geodesic on which g acts as a translation. In general, a hyperbolic isometry g does not need to possess an axis, however it does in case X is a pinched Hadamard manifold (they are complete CAT(0)-spaces). Or X is a Cayley graph of a word hyperbolic group Γ and $g = h^n$ for a constant $n = n(\Gamma)$ and an element h is of infinite order [Del96].

Lemma 3.1 ([CDP90], lemma 9.2.2.) *Suppose that for an isometry g of a $\delta-$hyperbolic space X there is a point $x \in X$ such that*

$$|g^2x - x| > |gx - x| + 2\delta.$$

Then g is hyperbolic.

Lemma 3.2 ([CDP90], lemma 9.2.3.) *Suppose that* $\delta > 0$. *Let* $g, h \in IsomX$ *be such that for some* $x \in X$ *the following holds*

$$\min\{|gx - x|, |hx - x|\} \geq 2(gx \cdot hx)_x + 6\delta.$$

Then gh, hg *are hyperbolic.*

Let S be a finite set of isometries of a δ-hyperbolic space X . The size of S at $x \in X$ is

$$|S|_x = \max_{s \in S} |sx - x|.$$

The following is a variant of Proposition 3.2 from [Kou98] for actions of a hyperbolic group.

Lemma 3.3 (Long generating system gives rise to a short hyperbolic isometry) *Let* X *be a geodesic* δ*-hyperbolic space and let* Γ *be a group of isometries of* X, *generated by a finite set of isometries* S. *Suppose that* $|S|_x > 100\delta$ *for each* $x \in X$. *Then there is a hyperbolic isometry in* Γ *which is a product of at most 2 isometries from* S.

Proof. ([Kou98]) Suppose that $|S|_x > 100\delta$ for each $x \in X$ and that S consists of nonhyperbolic isometries. Let x_0 be a point of almost minimal displacement for S that is

$$|S|_{x_0} \leq \inf_{y \in X} |S|_y + \delta.$$

Let $S_0 = \{a \in S : |ax_0 - x_0| \geq 50\delta\}$ be the long part of S. Fix $a_1 \in S$ such that $|a_1x_0 - x_0| = \max_{a \in S} |ax - x| > 100\delta$.

Case 1): *There is* $a \in S_0$ *such that* $(a_1x_0 \cdot ax_0)_{x_0} \leq 20\delta$. Applying the lemma 3.2 we obtain that aa_1 is hyperbolic.

Case 2): $(a_1x_0 \cdot ax_0)_{x_0} \geq 20\delta$ *for all* $a \in S_0$. We will show that this assumption leads to a contradiction of the almost minimality of x_0. Namely, we shall show that for the point x_1 on the geodesic $[x_0, a_1x_0]$ such that $|x_1 - x_0| = 20\delta$, the following holds

$$|S|_{x_1} < |S|_{x_0} - 10\delta;$$

and this contradicts the almost minimality of x_0.

Choose $x_a \in [x_0, ax_0]$ so that $|x_a - x_0| = 20\delta$, then, since $(a_1x_0 \cdot ax_0)_{x_0} \geq 20\delta$, by applying the inequality (3) to the triple of points (x_0, a_1x_0, ax_0), we have that the $|x_1 - x_a| \leq 4\delta$. Consider x_1 as a new base point and let $a \in S$.

We assert the following:

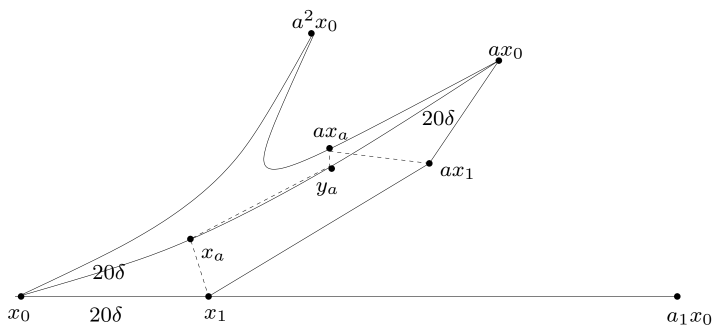

Figure 5: Long generating system gives rise to a uniform hyperbolic isometry.

- if $a \in S_0$, then $|ax_1 - x_1| \leq |ax_0 - x_0| - 28\delta$;
- if $a \notin S_0$, then $|ax_0 - x_0| \leq 90\delta$.

It follows from this assertion that

$$\max_{a \in S} |ax_1 - x_1| \leq \max\{\max_{a \in S} |ax_0 - x_0| - 28\delta, 90\delta\} < \max_{a \in S} |ax_0 - x_0| - 10\delta,$$

contradicting the almost minimality of x_0.

To prove the assertion, consider as a first case when $a \in S_0$. Choose $y_a \in [x_0, ax_0]$ so that $|y_a - ax_0| = 20\delta$. We have

$$(a^2x_0 \cdot x_0)_{ax_0} = \frac{1}{2}(|ax_0 - x_0| + |a^2x_0 - ax_0| - |a^2x_0 - x_0|) \geq \frac{1}{2}|ax_0 - x_0| - \delta \geq 24\delta. \tag{4}$$

Applying inequality (3) to the triangle $\Delta(x_0, ax_0, a^2x_0)$ we obtain $|ax_a - y_a| \leq 4\delta$. By assumption $(a_1x_0 \cdot ax_0)_{x_0} \geq 20\delta$ hence $|ax_a - ax_1| = |x_a - x_1| \leq 4\delta$.

We conclude that

$$\begin{aligned} |ax_1 - x_1| &\leq |ax_1 - ax_a| + |ax_a - y_a| + |y_a - x_a| + |x_a - x_1| \leq \\ & \quad 4\delta + 4\delta + (|ax_0 - x_0| - 40\delta) + 4\delta \\ &\leq |ax_0 - x_0| - 28\delta. \end{aligned}$$

For the second case when $a \notin S_0$ we have

$$\begin{aligned} |ax_1 - x_1| &\leq |ax_1 - ax_0| + |ax_0 - x_0| + |x_0 - x_1| \leq \\ 20\delta + 50\delta + 20\delta &\leq 90\delta \end{aligned}$$

□

4 Geometrically finite groups of isometries of Hadamard manifolds

An Hadamard manifold is a simply connected, complete Riemannian manifold without boundary and with nonpositive sectional curvatures. We will assume moreover that X is pinched, i.e. all the sectional curvatures lie between two negative constants: $-\kappa^2 \leq K(X) \leq -1$, [Bow95]. A convenient reference for such manifolds is [BGS85].

A basic fact about Hadamard manifold is that the exponential map based at any point is injective. Thus, any such manifold X is diffeomorphic to $\mathbb{R}^n$. X can be naturally compactified by adjoining an ideal sphere ∂X to X, so that $\overline{X} = X \cup \partial X$ is homeomorphic to a closed n-dimensional ball. A pinched Hadamard manifold is a visibility manifold, i.e. any two points $x, y \in \overline{X}$ can be joined by a unique geodesic, which we denote by $[x, y]$. When we speak of geodesic as paths, it will be assumed they are parameterized by arc-length. Denote by $|x - y|$ a Riemannian path distance between $x, y \in X$. The point $\xi \in \partial X$ can be represented by a geodesic ray $c_\xi : [0, \infty) \to X$. The function $h_c(x) = \lim_{t\to\infty}(|x - c(t)| - t)$ is called the horofunction about ξ associated to the geodesic ray c. It turns out that h is C^2 [HIH77] and the norm of its gradient is everywhere equal to 1. The level sets of h_c are called horospheres about ξ. The horospheres form a codimension 1 foliation of X orthogonal to the foliation by bi-infinite geodesics having one endpoint at ξ. A set of the form $h_\xi^{-1}[r, \infty)$ for $r \in \mathbb{R}$ is called a horoball about ξ. Such a horoball may alternatively be described as the closure of the set $\cup\{\{B_t(\beta(t)) : t \in (0, \infty)\}$ where β is a geodesic ray tending to ξ with $\beta(0) \in h_\xi^{-1}(r)$. In particular horoballs are convex.

The well known classification of isometries of a hyperbolic space holds for X too. Any isometry g of X extends to a homeomorphism of $\overline{X}$. We shall write $\overline{X}^g$ for the set of fixed points of g in $\overline{X}$. Any nonidentical isometry g of X is precisely on of the following types:

1) g is elliptic, that is X^g is nonempty,

2) g is parabolic, that is $\overline{X}^g$ consists of a single point $p \in \partial X$ and g preserves setwise each horosphere about p,

3) g is hyperbolic, that is $\overline{X}^g = \{p, q\}$, where p and q are distinct points of ∂X. In this case g translates along the axis - the geodesic connecting p and q and we denote by $||g||$ the amplitude of this translation (= translation length.

Note that this definition of hyperbolicity is consistent with the one given before for isometries of a δ- hyperbolic space.

By definition, the group $\Gamma \leq \text{Isom } X$ is elementary if $\overline{X}^{\Gamma} \neq \emptyset$ or else Γ preserves bi-infinite geodesic in $\overline{X}$.

Definition 4.1 Let X be a pinched Hadamard manifold and $\Gamma \leq \text{Isom } X$ is a subgroup of isometry group, acting properly. Γ is geometrically finite if the following properties hold:

- there is a Γ-invariant family (possibly empty) $\mathcal{B} = \{B_p \mid p \in P\}$ of pairwise disjoint closed horoballs in X with $\Gamma\backslash\mathcal{B}$ finite;
- there is a non-empty closed convex Γ-invariant subset $Z \subseteq X$ such that the Γ-action on the neutered space $X_\Gamma = Z - \bigcup_{p \in P} \text{Int}(B_p)$ is cocompact.

The more restrictive class of groups, consisting of lattices, corresponds to the case of $Z = X$ in the above definition.

We now return to the discussion of the geometry of X_Γ. Each maximal parabolic subgroup P of Γ fixes a point at infinity of X and hence fixes any horoball in X centered at this point. We consider $X_\Gamma = Z - \bigcup_{P \in \mathbb{P}} Int(B_P)$ with the path metric, that is the metric given by lengths of paths, computed using the standard hyperbolic Riemannian metric. X_Γ is complete and locally compact, so it is a geodesic metric space. Γ acts on X by isometries, with finite stabilizers, and with compact quotient.

We call the boundary piece $S_P = X \cap \partial B_P$ that results from removing $int(B_P)$ a horosphere of X. Two horospheres S_P and $S_{P'}$ are have the same image in M if and only if P and P' are conjugate in Γ.

Since Z is convex, the metric on Z is the restriction of the metric on X. In particular, it is a geodesic metric space, geodesics are unique, and they vary continuously with choice of endpoint. For points $x, y \in X$ we will denote $d_X(x, y)$ their distance apart in X and $d_{X_\Gamma}(x, y)$ their path distance in X_Γ. The path metric on X_Γ is at most exponentially distorted with respect to d_X; namely

$$d_X(x, y) \leq d_{X_\Gamma}(x, y) \leq \sinh(\kappa^2 \cdot d_X(x, y))$$

for all $x, y \in X$. This is proven in [Far95] for the case of lattices but the proof goes equally well for geometrically finite groups. Thus, there is a monotonically increasing positive function ϕ, does not depending on Γ such that $\phi(x) \to \infty$ with $x \to \infty$ and such that $d_X > \phi(d_{X_\Gamma})$.

By a result of Bowditch all P are virtually nilpotent [Bow95] (this follows from the Margulis Lemma).

5 Uniform hyperbolic elements in groups acting on Hadamard manifolds

In the case of noncocompact discrete actions we know of no results about uniform hyperbolic elements as proven in the previous section. But we have

Lemma 5.1 *Let Γ be a nonelementary geometrically finite group acting on a Hadamard manifold X. There is neutered space X_Γ and a constant $\Delta > 0$ such that if Γ is generated by the set S and $|S|_x^{X_\Gamma} > \Delta$ for each $x \in X$, then there is an hyperbolic isometry in Γ which is a product of at most 2 isometries from S. (The norm is taken relative to the path metric on X_Γ.)*

Proof. Take any $\Delta > 0$ such that $\phi(\Delta) > 100\delta$, where ϕ is defined in the previous section. Choose the horoball system so that that for any two of them $\phi(d_{X_\Gamma}(B_1, B_2)) > 100\delta$. Now take the generating set S so that $|S|_x^{X_\Gamma} > \Delta$ for each $x \in X_\Gamma$. Then $|S|_x^X > \phi(\Delta) > 100\delta, x \in X_\Gamma$. To apply the Lemma 3.3 above it is enough to ensure that $|S|_x^X > 100\delta$ for all $x \in X$. We have already done this for $x \in X$. Now suppose x belongs to some horoball B . If $Sx \subset B$, then Γ preserves the horoball B and hence is elementary - contradiction. Thus $sx \notin B$ for some $s \in S$. The segment $[x, sx]$ crosses the horospheres $\partial B, s\partial B$ in some points x_1, x_2 and we have $|sx - x|_X \geq |x_2 - x_1|_X \geq \phi(|x_2 - x_1|_{X_\Gamma}) \geq \phi(d_{X_\Gamma}(B, sB)) > 100\delta$. Thus, we can apply the Lemma 3.3. □

Theorem 5.2 *A geometrically finite group Γ, acting on a pinched Hadamard manifold X, uniformly contains hyperbolic elements.*

Proof. Let X be a corresponding neutered space, obtained by removing from the convex hull of the limit set the disjoint Γ-invariant union of horoballs about parabolic fixed points. By Lemma 5.1 there is a $\Delta > 0$ such that if Γ is generated by S and $|S|_x^{X_\Gamma} > \Delta$ for each $x \in X_\Gamma$, then there is a hyperbolic isometry in Γ which is a product of at most 2 isometries from S. (The norm is taken relative to the path metric on X_Γ.) Denote by $\mathcal{G}_0$ the set of all finite generating sets S for Γ such that $|S|_x^{X_\Gamma} \leq \Delta$ for some $x \in X_\Gamma$. Since the action on X_Γ is proper and cocompact, the set $\mathcal{G}_0$ is finite up to conjugacy. Denote by $\mathcal{G}_1$ the set of all finite generating sets S for Γ do not belonging to $\mathcal{G}_0$. We want to get a universal bound for the length of a shortest hyperbolic element in an arbitrary generating system. Firstly it is easy to do this for

$\mathcal{G}_0$. Indeed, this length is invariant under conjugation, and $\mathcal{G}_0$ is finite up to conjugacy. Next, if $|S|_x^{X_\Gamma} > \Delta$ for each $x \in X$, then by the proposition above there is a hyperbolic isometry in Γ which is a product of not more than 2 isometries from S. $\square$

6 Separation of axes

Definition 6.1 By an axis of an isometry g of a metric space X we call in isometric copy of $\mathbb{R}$ inside X on which g acts by translation. We say that the group Γ of isometries of a metric space X satisfies $SA-$property (= Separation of axes) if for every $\varepsilon > 0$ there is $b(\varepsilon) \geq 0$ such that for any $g, h \in \Gamma$ possessing the axes and having the same translation length either their axes are asymptotic (=fellow travel each other) or the following inequality holds: $\ell(\sigma_g, \sigma_h, \varepsilon) < b(\varepsilon)a$, where σ_g, σ_h are the axes of g, h respectively, and a is the amplitude of translation of g, h on the axes. (See 2.2 for definition of ℓ).

Lemma 6.2 (Separation of axes) *If the group Γ acts as a geometrically finite group of isometries of a pinched Hadamard manifold X, then Γ satisfies the SA-property.*

Proof. For $r > 0$ denote by b_r the maximum of cardinalities of the sets S of elements of Γ, such that some orbit $Sx, x \in X_\Gamma$ is contained in the ball of radius r. Since the action of Γ on X_Γ is cocompact, b_r is finite. We assert that the SA-property is satisfied with $b(\varepsilon) = b_\varepsilon + 1$.

Suppose the contrary then there is $\varepsilon > 0$ and hyperbolic isometries $g, h \in \Gamma$ such that $\ell(\sigma_g, \sigma_h, \varepsilon) > (b_\varepsilon + 1)a$, where σ_g, σ_h are axes of g, h and a is an amplitude of translation of g, h. Denote by σ'_g, σ'_h the subsegments of σ_g, σ_h respectively such that $|\sigma'_g(t) - \sigma'_h(t)| \leq \varepsilon, 0 \leq t \leq T$, and $T \geq (b_\varepsilon + 1)a$. Note that σ_g cannot stay within any horoball $B \in \mathcal{B}$, for time longer than a since otherwise g would preserve B and hence would be parabolic. Hence, cutting T by amount a we may assume that $|\sigma'_g(t) - \sigma'_h(t)| \leq \varepsilon, 0 \leq t \leq T$, $T \geq b_\varepsilon a$ and $\sigma'_h(0) \in X_\Gamma$. Changing if necessarily g, h by their inverses, we may assume that the action of g, h is coherent with the natural orientations of σ'_g, σ'_h. Then $g^i\sigma'_g(0) = \sigma'_g(ia) \in \sigma'_g, h^i\sigma'_h(0) = \sigma'_h(ia) \in \sigma'_h, i = 0, 1, \ldots, b_\varepsilon$ and we conclude that $|g^i\sigma'_g(0) - h^i\sigma'_h(0)| \leq \varepsilon, i = 0, 1, \ldots, b_\varepsilon$, hence $|\sigma'_g(0) - g^{-i}h^i\sigma'_h(0)| \leq \varepsilon, i = 0, 1, \ldots, b_\varepsilon$. Thus we have $b_\varepsilon + 1$ elements $g^{-i}h^i\sigma'_h(0), i = 0, 1, \ldots, b_\varepsilon$, inside the ball of radius ε about $\sigma'_h(0)$. All of these elements lie in X_Γ, since by our construction $\sigma'_h(0) \in X_\Gamma$! By definition of b_ε, at least two of these elements coincide, say $g^{-i}h^i\sigma'_h(0) = g^{-j}h^j\sigma'_h(0), 0 \le i \neq j \le b_\varepsilon$. We conclude that $g^{i-j} = h^{i-j}$ for some $1 \le i \neq j \le b_\varepsilon$. It follows that $|\sigma_g(n(i-j)) - \sigma_h(n(i-j))| \le \varepsilon$ for all integral n and hence the axes fellow travel each other. Hence they coincide by uniqueness of geodesics. □

7 Translation discreteness

It is known that if Γ is a group of isometries acting properly and cocompactly on a convex (in particular CAT(0)) metric space M then Γ is translation discrete in the sense that translation numbers of its nontorsion isometries are bounded away from zero, [Con00], Corollary 3.8.

We generalize this to geometrically finite groups

Theorem 7.1 *Suppose that the group Γ acts as a nonelementary geometrically finite group of isometries of a pinched Hadamard manifold X. Then the action is translation discrete in a sense that the translation numbers of its hyperbolic isometries are bounded away from zero.*

Proof. Let $\mathcal{B}$ be the horoball system attached to Γ acting on X and X_Γ - the corresponding neutered space. Decreasing horoballs from $\mathcal{B}$ we may assume that that the distances between two distinct horoballs are bounded away from zero. Otherwise, there is a sequence of hyperbolic elements g_i with translation numbers tending to 0. Consider first the case when all the axes A_i avoid $\mathcal{B}$. Fix any $x_i \in A_i \subset X_\Gamma$, then $||g_i x_i - x_i|| \to 0$ with $i \to \infty$. Since $\Gamma \backslash X_\Gamma$ is compact there is a compact subset $D \subset X_\Gamma$ such that $\Gamma D = X_\Gamma$. For every i there is $h_i \in \Gamma$, so that $h_i x_i \in D$. Choosing subsequence we may assume that $h_i x_i \to x \in D$. Then we have $|h_i g_i h_i^{-1} x - x| \le |h_i g_i h_i^{-1} x - h_i g_i h_i^{-1} h_i x| + |h_i g_i h_i^{-1} h_i x - h_i x| + |h_i x - x| = 2|h_i x - x| + |h_i g_i x - h_i x| = 2|h_i x - x| + |g_i x - x| \to 0, i \to \infty$. This contradicts to the properness of Γ.

Now consider the case when there are infinitely many axes A_i meeting $\mathcal{B}$ - we may then assume that all of them meet $\mathcal{B}$. Since each g_i translates nontrivially each horoball it visits, the translation length of g_i is at least as large as the distance between distinct horoballs; thus it is bounded away from zero - a contradiction. □

8 Free subgroups

Koubi's criterion for free subgroups is as follows. We generalize it to our situation.

Lemma 8.1 (Freeness criterion)*([Kou98], Lemma 2.4) Let g, h be the isometries of a*
$\delta-$hyperbolic space X. Suppose that for a base point x_0 the following holds:

$$|g^{\pm 1}x_0 - h^{\pm 1}x_0| > \max(|gx_0 - x_0|, |hx_0 - x_0|) + 2\delta,$$

$$|g^2x_0 - x_0| > |gx_0 - x_0| + 2\delta,$$
$$|h^2x_0 - x_0| > |hx_0 - x_0| + 2\delta.$$

Then g, h freely generate the free group F_2.

□

Lemma 8.2 *Let X be a $\delta-$hyperbolic space and let $\Gamma \leq$ Isom X be a nonelementary group acting properly on X. Suppose that the action Γ on X is translation discrete in the sense of Theorem 7.1 and satisfies the Separation Axes Property. Then there is a constant $m = m(\Gamma, X)$ such that for every hyperbolic $g_0 \in \Gamma$ having axis and for any generating set S there exists $s \in S$ such that $g = g_0^m$ and $h = s^{-1}gs$ freely generate a free group of rank 2.*

Proof. Since Γ on X is translation discrete there is a constant $c_0 > 0$ such the translation length of any hyperbolic element, possessing an axis, is greater than c_0. Further, let $b_0 = b(372\delta)$ be the constant given by the SA-property. We assert that $m > \frac{1}{c_0}722\delta + 2b_0$ is big enough to satisfy conclusion of the lemma. Suppose $g_0 \in \Gamma$ has an axis σ_g and let S be a finite generating system for Γ. Since Γ is nonelementary, there exists $s \in S$ not fixing the pair $\{\sigma(-\infty), \sigma(\infty)\}$. Then the axes $\sigma_{g_0}, \sigma_{h_0}$ of $g_0, h_0 = sg_0s^{-1}$ have disjoint limit sets. By Separation Axes Property

$$\ell_0 = \ell(\sigma_{g_0}, \sigma_{h_0}, 380\delta) \leq b_0||g_0||. \tag{5}$$

By the choice of m the translation length $m||g_0||$ of $g = g_0^m$ is greater than $722\delta + 2b_0||g_0||$. Reparameterize the geodesics so that at the moment $t = 0$ they are located at the points $x_0 \in \sigma_g = \sigma_{g_0}, y_0 \in \sigma_h = \sigma_{h_0}$, which are the closest ones among the points of these axes.

We shall verify conditions of the criterion of freeness 8.1. We start with the condition

$$|gx_0 - hx_0| > \max\{|gx_0 - x_0|, |hx_0 - x_0|\} + 2\delta.$$

Remote case: Suppose that $|x_0 - y_0| = |hx_0 - hy_0| > 180\delta$. The 5-gon $[gx_0, x_0, y_0, hy_0, hx_0]$ satisfies assumptions of Lemma 2.5. Indeed, the lengths of the sides $[gx_0, x_0], [hy_0, hx_0]$ are at least 180δ each. Angles at the points x_0, y_0, hy_0 are all obtuse, since, for example, x_0 is the nearest to y_0 point on σ_g. Hence by Lemma 2.4 $(gx_0 \cdot y_0)_{x_0} < 8\delta \leq 14\delta$.

We conclude that

$$|gx_0 - hx_0| > |gx_0 - x_0| + |hy_0 - y_0| + 2|x_0 - y_0| - 168\delta \geq$$
$$|gx_0 - x_0| + |hx_0 - x_0| - 168\delta.$$

Applying lemma 2.5 to the 4-gon $[x_0, y_0, hy_0, hx_0]$ we obtain

$$|hx_0 - x_0| > 2|x_0 - y_0| + |hy_0 - y_0| - 168\delta > 180\delta,$$

hence, combining with the above we obtain

$$|gx_0 - hx_0| > \max\{|gx_0 - x_0|, |hx_0 - x_0|\} + 2\delta.$$

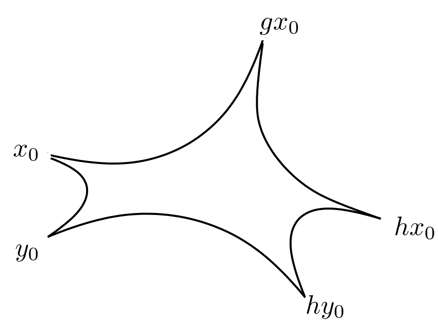

Figure 6: Geodesic 5-gon.

Nearby case: Suppose that $|x_0 - y_0| = |hx_0 - hy_0| \leq 180\delta$. Then by lemma 2.6 applied to the 5-gon $[gx_0, x_0, y_0, hy_0, hx_0]$ we have

$$|gx_0 - hx_0| > |gx_0 - x_0| + |hy_0 - y_0| - 360\delta - 2b_0||g_0|| \geq$$

(since both summands below are $\geq 722\delta + 2b_0||g_0||$)

$$\max\{|gx_0 - x_0|, |hy_0 - y_0|\} + 362\delta \geq$$

(since $|hy_0 - y_0| \geq |hx_0 - x_0| - 360\delta$))

$$\max\{|gx_0 - x_0|, |hx_0 - x_0|\} + 2\delta.$$

It remains to verify the conditions $|g^2x_0 - x_0| > |gx_0 - x_0| + 2\delta$, $|h^2x_0 - x_0| > |hx_0 - x_0| + 2\delta$.

The first is clear since $|g^2x_0 - x_0| = 2|gx_0 - x_0|$. For the second, since x_0 does not lie on the axis of h, we shall consider the remote and nearby cases. If $|x_0 - y_0| = |hx_0 - hy_0| > 180\delta$ then applying Lemma 2.5 to the 4-gon $[x_0, y_0, h^2y_0, h^2x_0]$ we obtain

$$|h^2x_0 - x_0| > 2|x_0 - y_0| + |h^2y_0 - y_0| - 168\delta = 2|x_0 - y_0| + 2|hy_0 - y_0| - 168\delta$$

$$> 360\delta + |hy_0 - y_0| - 168\delta > |hy_0 - y_0| + 2\delta.$$

In the nearby case we have $|x_0 - y_0| = |hx_0 - hy_0| \leq 180\delta$ and then

$$|h^2x_0 - x_0| \geq |h^2y_0 - y_0| - 2|x_0 - y_0| \geq 2|hy_0 - y_0| - 360\delta > |hy_0 - y_0| + 360\delta.$$

□

Proof of the theorem 1.1. Suppose that Γ is a nonelementary geometrically finite group of isometries of a pinched Hadamard manifold X. Denote by X_Γ the corresponding neutered space. Since Γ acting on X_Γ is proper and cocompact it satisfies Separation Axes Property. By Lemma 6.2 Γ acting on X satisfies Separation Axes Property too. Moreover by Theorem 5.2, Γ uniformly contains hyperbolic elements. Hence by lemma 8.2 Γ uniformly contains a free group of rank two. □

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Roger C. Alperin:
Dept. of Mathematics,
San Jose State University
San Jose, CA 95192
E-mail: alperin@math.sjsu.edu

Gennady A. Noskov:
Institute of Mathematics,
Russian Academy of Sciences,
Pevtsova 13, Omsk, 644099, Russia
and
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131,
D-33501 Bielefeld,
Germany
E-mail: noskov@mathematik.uni-bielfeld.de