AXIOMS FOR ORIGAMI AND COMPASS CONSTRUCTIONS

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1. Introduction

We describe the axioms of a single fold origami system where, in addition to the usual Huzita-Justin axioms, one may also use a compass to create circles. An axiom of this type was implicitly used by [Edwards and Shurman 2001], which provided as fold lines the common tangents to a circle and parabola. (We say a circle is *compass constructible* iff its center and an incident point are known.) This allows one to construct the roots to the general quartic polynomial equation. Recently, this idea of constructions using a circle together with origami has been pursued by [Kassem, Ghourabi and Ida 2011]. They added three axioms to the usual Huzita-Justin axioms for single fold origami, obtaining a system which also is *not* more powerful than single fold origami. However with the addition of their axioms one can perform some constructions in an nice way, as they show by implementing a classical method for the trisection of an angle. In the final section, we give two additional methods for trisection of an angle which are similar to ancient techniques based on *neusis*.

In this article we complete these systems by considering the full range of axioms for constructions with compass and single fold origami. We allow foldings which align by reflection across the fold, any of the geometric objects, either point, line or circle with another such object. For us aligning will mean incidence when a point is involved or in the case of two curves, an alignment is a tangency.

There are 22 axioms in all; the new axioms labelled O_{5b}, O_{6b}, O_{7b} are those of [Kassem, Ghourabi and Ida 2011]. Although our system is not more powerful than ordinary origami, it does allow one to elegantly fold the tangents to a given conic from a given point or the common tangents to two conics, thus resolving a question posed in [Alperin 2002].

2. Geometry of the basic folds

2.1. **Basic Folds.** We use P to denote a point, L to denote a line, and K denotes a circle. The basic folds are the following single alignments: $P_1 \leftrightarrow P_2$; $P_1 \leftrightarrow L_1$; $P_1 \leftrightarrow K_1$; $L_1 \leftrightarrow L_2$; $L_1 \leftrightarrow K_1$; $K_1 \leftrightarrow K_2$. The objects in the basic fold alignment are not necessarily distinct. The arrow means that the first object is reflected (transformed) across an origami fold line so that it is tangent (incident) to the second object or similarly the second object is transformed across the fold line to be incident with the first; in this way $\leftrightarrow$ is a symmetric operation. In the case of a point, tangency means incidence; whereas the tangency afforded by $L_1 \leftrightarrow L_2$ means that the folded image L'_1 of the line L_1 is coincident with the line L_2.

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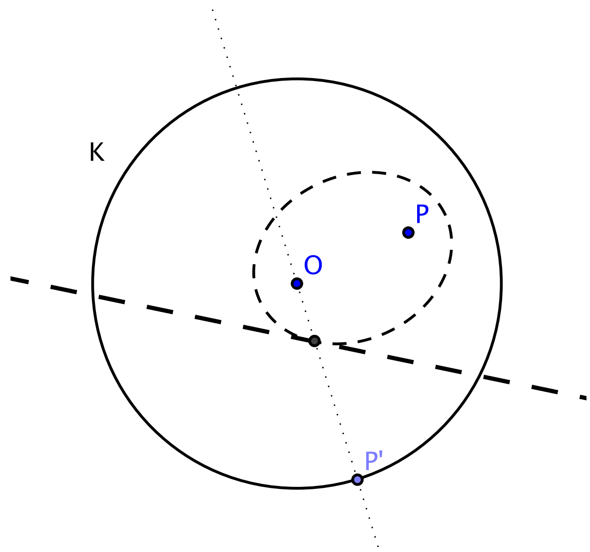

FIGURE 1. Focal Conic: $P \leftrightarrow K$

A basic fold where the elements are equal is called of *fixed* type, otherwise it is *generic* type.

Here are descriptions of the basic folds. In order to describe the axioms (see below), it is important to distinguish the geometry of all possible folds for the fixed type and generic type. In our description here we assume that distinctly labelled elements are different.

2.2. $P_1 \leftrightarrow P_2$**.** The fold is the perpendicular bisector of P_1P_2.

2.3. $P \leftrightarrow P$**.** These folds are all the lines through P.

2.4. $P \leftrightarrow L$**.** These folds form the envelope of tangents to the parabola with focus P and directrix L.

2.5. $P \leftrightarrow K$**.** We consider the envelope of the perpendicular bisectors of the given point P and a variable point P' on a circle K. These bisector lines are the origami folds $P \leftrightarrow K$. These lines form the envelope of tangents to a conic. The conic is an ellipse interior to the circle if the given point is interior to the circle, or a hyperbola if the point is exterior to the circle. The foci of the conic are the given point P and the center F of the circle. This is a familiar classical construction. See [Lockwood 1963]. If P lies on K then the fold line passes through the center of K or is tangent to K at P. Since these are readily constructed with ordinary origami folds we shall assume P is not on K.

2.6. $L_1 \leftrightarrow L_2$**.** The folds are the angle bisectors of the lines if the lines are not parallel; if the lines are parallel then the fold line is the midline of the two given lines.

2.7. $L \leftrightarrow L$**.** These folds are all the lines perpendicular to the given line L. The fold line L is also an admissible reflection, taking L to itself, but we ignore this case since no new object is created.

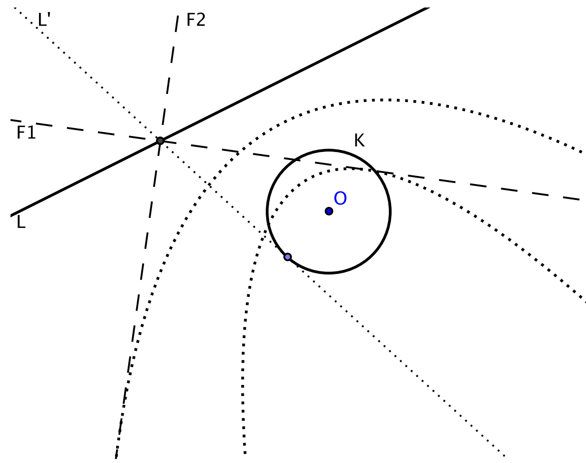

Figure 2. $L \leftrightarrow K$

2.8. $L \leftrightarrow K$. The possible fold lines F_1, F_2 are the bisectors of the line L with each of the tangents of K. These bisectors give two families, tangent to two parabolas with the focus O, the center of K. The directrices are parallel to the given line L at distance equal to the radius of K. The line through O and the point of tangency on K meets the folds in the points of tangency of the parabolas.

2.9. $K_1 \leftrightarrow K_2$. When K_1 is folded to K_2 it is either tangent as interior circle K_1^- or exterior circle K_1^+ to K_2. The center of K_1 is O, and the centers of the image circles are respectively O^- and O^+. The reflections or folds achieving these tangencies are the perpendicular bisectors of O and O^- or O and O^+. These family of lines envelope conics $\mathcal{K}^-$, $\mathcal{K}^+$ which are either hyperbolas or ellipses with the foci O and the center U of K_2. The conics are either: ellipses if K_1 is interior to K_2; hyperbolas if K_1 is exterior to K_2; otherwise an ellipse and hyperbola.

2.10. $K \leftrightarrow K$. A fold $K \leftrightarrow K$ can be internally tangent; in this case the only possible folds are lines through the center P; these folds through the center P are the same as $P \leftrightarrow P$. We shall replace these folds of K to itself by the simpler $P \leftrightarrow P$. However if the transform of K is externally tangent to K then the fold is a tangent line of K; as in the discussion above the envelope is the conic K. We can also realize these tangent lines by folding the center of the circle P to a circle of twice the radius K_1, i.e. $P \leftrightarrow K_1$.

So we can absorb this alignment into others and do not need to consider it.

3. Axiom Types

Axioms are sets of these basic folds which have only finitely many possible origami fold line solutions; so we combine the basic folds to obtain axioms.

As we see from the discussion above there are only finitely many folds for $P_1 \leftrightarrow P_2$, $L_1 \leftrightarrow L_2$ and in the remaining cases the solutions form an algebraic curve of small degree in the projective space of lines. Thus by imposing at most two of these conditions we will have only finitely many solutions. These form the set of axioms.

The axioms O_1, O_2, O_{3a}, O_{4a}, O_{5a}, O_{6a}, O_{7a} (see below) are defined by the standard one fold origami alignments.

In the table below there are 15 circle-origami axioms using two basic folds. We do assume that the column element is distinct from the row element. However distinctly labelled circles may be equal since the alignment generates an algebraic curve. For example an instance of $3d$, $K \leftrightarrow K_1, K \leftrightarrow K_2$, would represent the simultaneous alignment of K, K_1, K_2; there are only finitely many possibilities as long as $K_1 \neq K_2$.

	$P_2 \leftrightarrow K_2$	$L_2 \leftrightarrow K_2$	$K_3 \leftrightarrow K_4$
$P_1 \leftrightarrow P_1$	5b	4b	4c
$P_1 \leftrightarrow L_1$	6b	7c	7e
$P_1 \leftrightarrow K_1$	6c	7d	7f
$L_1 \leftrightarrow L_1$	7b	8a	8b
$L_1 \leftrightarrow K_1$		3b	3c
$K_1 \leftrightarrow K_2$			3d

3.1. **Eight Axiom Types.** We use a numbering system consistent with the standard Huzita-Justin axioms. The axioms are exhaustively described as one of the following types:

(1) No points are transformed: this is either O_3, or O_8, where the first has no fixed curves, the second has one fixed curve.
(2) Only one point is transformed and it is fixed: this is O_4.
(3) Only one point is transformed and it is not fixed: this is O_7.

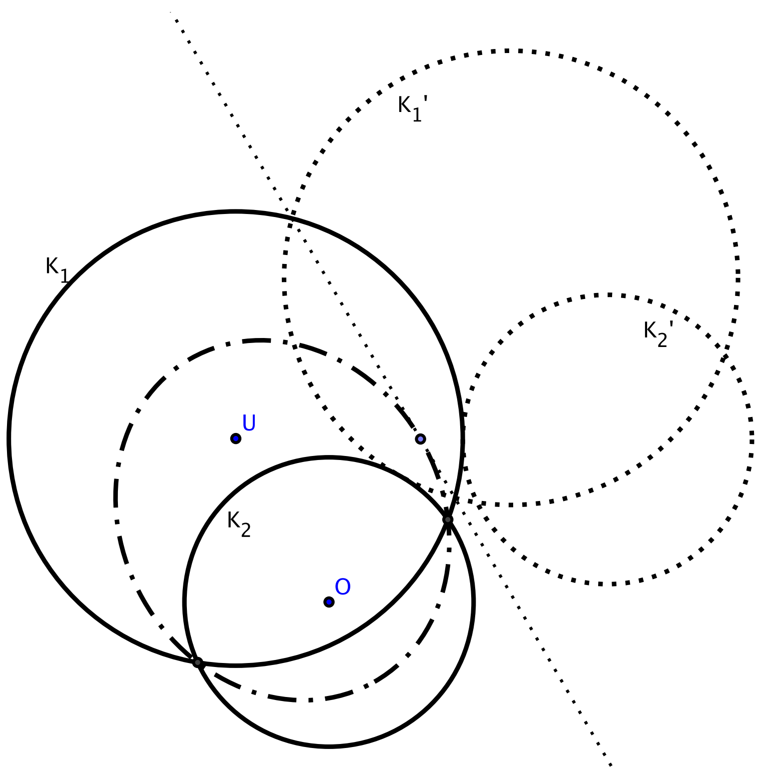

FIGURE 3. $K_1 \leftrightarrow K_2$

(4) Two points are transformed and both are fixed: this is O_1
(5) Two points are transformed and one is fixed: this is O_5.
(6) Two points are transformed and neither is fixed: if there are no curves this is O_2, otherwise O_6.

4. Axioms

4.1. O_1**:** $P_1 \leftrightarrow P_1$, $P_2 \leftrightarrow P_2$.
The fold line is incident with the given points.

4.2. O_2**:** $P_1 \leftrightarrow P_2$.
The fold line is the perpendicular bisector of the given points.

4.3. O_3**:**
a) $L_1 \leftrightarrow L_2$. The fold line is a bisector of the given lines. There are two solutions.
b) $L_1 \leftrightarrow K_1$, $L_2 \leftrightarrow K_2$ The folds are the common lines to a pair of conic envelopes. There are at most 8 solutions, but the algebraic degree is at most 4 (factors into two quartics).
c) $L_1 \leftrightarrow K_2$, $K_3 \leftrightarrow K_4$ The folds are the common lines to four different pairs of conic envelopes. Thus there are at most 16 solutions, but the algebraic degree is at most 4.
d) $K_1 \leftrightarrow K_2$, $K_3 \leftrightarrow K_4$ The folds are the common lines to four different pairs of conic envelopes. Thus there are at most 16 solutions, but the algebraic degree is at most 4.

4.4. O_4**:**
a) $P_2 \leftrightarrow P_2$, $L_1 \leftrightarrow L_1$ The fold line is incident with the point and perpendicular to the line. There is one fold solution.
b) $P_1 \leftrightarrow P_1$, $L_2 \leftrightarrow K_2$ The fold line is incident with the point and reflects the given line to a tangent of the circle. The reflections of the given tangent to the circle has a bisector with the given line enveloping a parabola. Thus there are at most two fold passing through a given point.
c) $P_2 \leftrightarrow P_2$, $K_1 \leftrightarrow K_2$. The fold line is incident with the given point and belongs to one of two possible confocal conic envelopes. Thus there are 4 possible solutions, but the algebraic degree is at most 2.

4.5. O_5**:**
a) $P_2 \leftrightarrow L_2$, $P_1 \leftrightarrow P_1$. The fold line passes through P_1 and is part of the tangent envelope to a parabola. There are at most two possible solutions.
b) $P_2 \leftrightarrow K_2$, $P_1 \leftrightarrow P_1$. The fold line passes through P_1, and conic envelope of tangents to an ellipse or hyperbola. There are at most two possible solutions.

4.6. O_6**:**
a) $P_1 \leftrightarrow L_1$, $P_2 \leftrightarrow L_2$. There are in general at most three solutions given by the common tangents to two parabolas, [Alperin 2000].
b) $P_1 \leftrightarrow L_1$, $P_2 \leftrightarrow K_2$. The folds are the common tangents to a parabola and an ellipse or hyperbola. There are at most 4 solutions. The algebraic degree is 4.
c) $P_1 \leftrightarrow K_1$, $P_2 \leftrightarrow K_2$. These folds are the common tangents to two conics, so there are at most 4 solutions and the algebraic degree is 4.

4.7. O_7**:**

a) $L_1 \leftrightarrow L_1$, $P_2 \leftrightarrow L_2$. The fold line is perpendicular to L_1 and is tangent to a parabola. There is at most one solution.

b) $L_1 \leftrightarrow L_1$, $P_2 \leftrightarrow K_2$. The fold line is perpendicular to L_1 and tangent to an ellipse or hyperbola. There are at most two solutions.

c) $L_1 \leftrightarrow K_1$, $P_2 \leftrightarrow L_2$. The fold line is a common tangent to two different pairs of parabola envelopes. There are at most 6 solutions but the algebraic degree is 3.

d) $L_1 \leftrightarrow K_1$, $P_2 \leftrightarrow K_2$. The fold line is a common tangent to one of the common focus parabolas and confocal conic envelopes. There are 16 possible solutions but the algebraic degree is at most 4.

e) $K_1 \leftrightarrow K_2$, $P_2 \leftrightarrow L_2$. The fold line is a tangent to a one of the two parabola and confocal conic envelopes. There are at most 8 solutions, but the algebraic degree is 4.

f) $K_3 \leftrightarrow K_4$, $P_1 \leftrightarrow K_1$. The folds are common to paired conic envelopes. There are at most 8 solutions but the algebraic degree is 4.

4.8. O_8**:**

a) $L_1 \leftrightarrow L_1$, $L_2 \leftrightarrow K_2$. The fold line is perpendicular to the first line and tangent to one of two parabolas. There are at two solutions. The algebraic degree is 2.

b) $L_1 \leftrightarrow L_1$, $K_3 \leftrightarrow K_4$. The fold line is perpendicular to the first line and tangent to one of two conics. There are at most 4 solutions and the algebraic degree is 2.

4.9. **Removing the Operation** $L \leftrightarrow K$**.** Since the basic fold $L \leftrightarrow K$ gives the tangents to two different parabolas, in a quite similar way to the fold $P \leftrightarrow L$, we could easily remove that operation and get a simpler set of axioms as shown below, a total set of 16 axioms.

	$P_2 \leftrightarrow K_2$	$K_3 \leftrightarrow K_4$
$P_1 \leftrightarrow P_1$	5b	4c
$P_1 \leftrightarrow L_1$	6b	7e
$P_1 \leftrightarrow K_1$	6c	7f
$L_1 \leftrightarrow L_1$	7b	8b
$K_1 \leftrightarrow K_2$		3d

5. Constructions

We call an ellipse or hyperbola a focal conic. These axioms give constructions which solve some problems posed in [Alperin 2002]. In words these new axioms allow the constructions of the following.

O_{3d} : Common tangents to two given confocal (same foci) sets of conics.
O_{4d} : Tangents to two given confocal conics which pass through a given point.
O_{5b} : Tangents to a given focal conic through a given point.
O_{6b} : Common tangents to a given focal conic and parabola.
O_{6c} : Common tangents to a two given focal conics.
O_{7c} : Common tangents to two given pairs of parabolas.
O_{7d} : Tangents to given parabola which is tangent to another focal conic.
O_{7f} : Common tangents to a given focal conic and either of a pair of confocal conics.
O_{8b} : Tangents to confocal conics which are perpendicular to a given line.

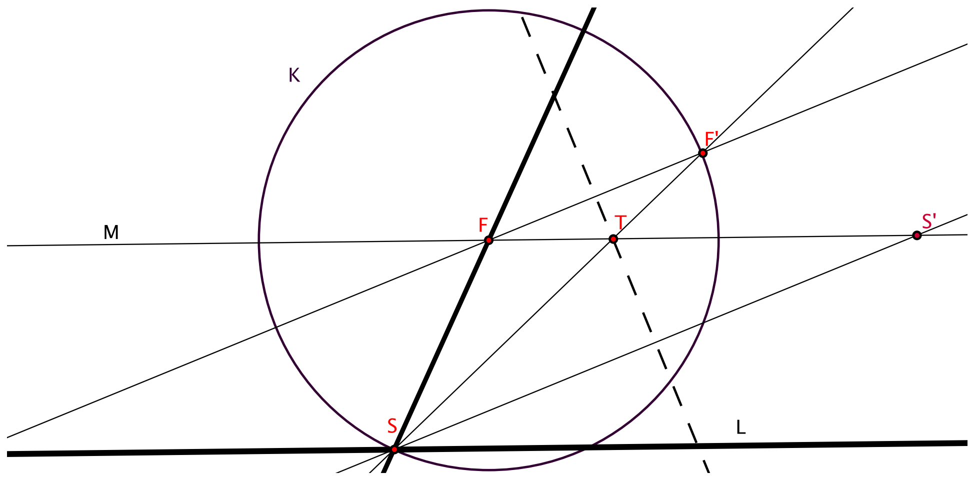

FIGURE 4. Trisection by O_{6b}

5.1. **Angle Trisection:** O_{6b}**.** Here is a classical construction (Jordanus, Campanus, 13th century) for the trisector of $\angle FSL$ with vertex S and sides FS and line L modified so that it is done by compass-origami. This is closely related to Archimedes trisection [Alperin 2005].

Let M be the line parallel to L through F. Construct the circle K centered at F with radius $|FS|$.

Fold $F \leftrightarrow K, S \leftrightarrow M$ using axiom O_{6b} (the focal conic is a circe). Let S', F' be the image by reflection across the fold line. The fold line meets M at T. By isosceles triangles and opposite angles cut on parallels $\angle TSS' = \angle TS'S = \angle TFF' = \angle TF'F = \alpha$. So $\angle F'TS' = \angle FTS = 2\alpha$. Then using parallels L, M cut by ST we have $\angle TSL = 2\alpha$ so $\angle S'SL = \alpha$. Also since SFF' is isosceles then $\angle FSF' = \alpha$ and thus $\angle FSL$ is trisected by $S'S$.

5.2. **Another Angle Trisection:** O_{6b}**.** Here is a another classical construction (Archimedes, 3rd century BC) for the trisector of $\angle FSL$ with vertex S and sides FS and line L; the construction has been modified so that it is done by compass-origami. It is also related to an origami trisection of Justin [Justin 1989].

Let N be the line perpendicular to L through S. Construct the circle K centered at S with radius $|FS|$.

Fold $S \leftrightarrow K, F \leftrightarrow N$ using axiom O_{6b}. There are in general 4 common tangents to the parabola with focus F and directrix N with the circle K_1 concentric with K with half the radius. These common tangents are the possible fold lines; one of the tangents, the perpendicular bisector of SF, is not useful for trisection (trivial case) .

Let S', F' be the image by reflection across a non-trivial fold line. $\angle FSS' = \angle SF'Q = \beta$, $\angle S'SL = \gamma$, $\beta + \gamma = \pi/2$. Therefore $\angle F'QS = \gamma$. Hence $\angle F'RS = \angle F'S'S = \angle FSS' = \alpha$. So $\alpha = 2\gamma$ by exterior angle of $\triangle RQS$ and therefore the acute $\angle FSL$ is trisected.

5.3. **Cube Root:** O_{3b}**.** Consider a circle of radius one, K, centered on the y-axis Y at $U = (0, -1)$, and a circle of radius a, K_a, centered on the x-axis X at

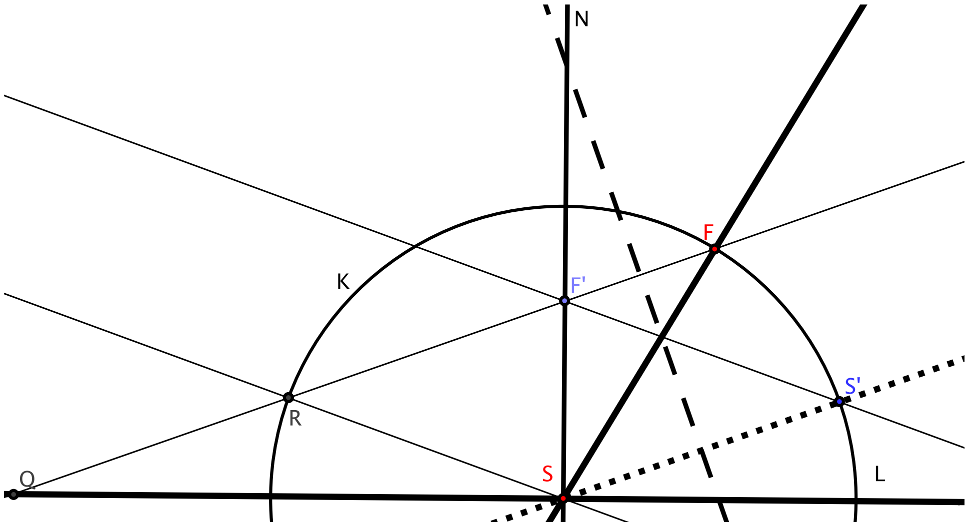

FIGURE 5. Another Trisection by O_{6b}

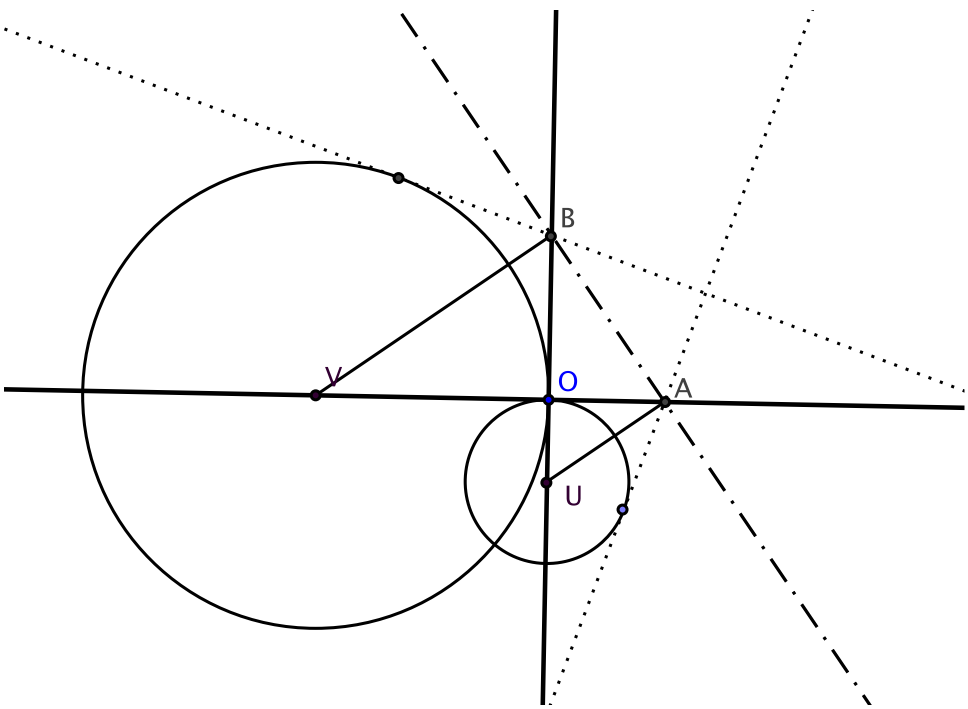

FIGURE 6. Cube Root by O_{3b}

$V = (-a, 0)$, hence tangent to the respective axes as in Figure 6. The we fold using axiom $3b$, $Y \leftrightarrow K_a, X \leftrightarrow K$. The line achieving this folding meets the axes at points A, B respectively.

Now one easily sees that there are 3 similar right triangles $\triangle OAU$, $\triangle OVB$, $\triangle OBA$. Using the common ratio of the legs we have $a \cdot OA = OB^2$ and $OB = OA^2$. Hence $a = OA^3$.

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