2-Colorings of Cube Edges With 6 Each

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This study was motivated by a problem posed by C. Morrow in her edge-colored cube constructions by origami.

1. Polya Counting

The enumerator for all 2-colorings the edges of the cube where we use b of color B and w of color W is the coefficient of B^bW^w in

$$B^{12} + B^{11}W + 5B^{10}W^2 + 13B^9W^3 + 27B^8W^4 + 38B^7W^5 +$$
$$48B^6W^6 + 38B^5W^7 + 27B^4W^8 + 13B^3W^9 + 5B^2W^{10} + BW^{11} + W^{12}.$$

Thus there are 218 different 2-colorings of the cube edges (sum the coefficients). The term $38B^5W^7$ means that there are 38 colorings with 5 B edges and 7 W edges. To obtain this formula we use the Polya counting theorem: substitute $x_i = B^i + W^i$ into the cycle index enumerator for the rotation group $\mathbf{O}^+$ of the cube acting on the edges

$$P_{\mathbf{O}^+} = \frac{1}{24}(x_1^{12} + 6x_4^3 + 3x_2^6 + 6x_1^2x_2^5 + 8x_3^4)$$

This cycle index formula is an average of the enumeration of the different cycle structures for the group G. For example there are 6 permutations whose cycle structure has two 1-cycles and five 2-cycles (center of opposite edge rotations).

There are 48 2-colorings of the edges of the cube with 6 of each color B, W. We can switch colors B, W to get another coloring, but some colorings are equivalent (after a rotation) to their color reverses-we call these symmetric colorings. There are 10 of these using a refined formula [**1**, 5.38]: evaluate the enumerator with $x_{odd} = 0, x_{even} = 2$ to obtain a total of 114 different 2-colorings without regard to the selection of the colors. Thus it follows that there are 10 symmetric colorings (sym) since

$$sym + 2(114 - sym) = 218$$

and all sym 2-colorings must be of the type having 6 colors each. Hence there are 29 different 2-colorings without regard to the choice of colors: $48 = 10 + 2(u - 10)$ has solution $u = 29$.

2. Symmetries of the Cube

There are 48 symmetries of the cube allowing for both orientation preserving and reversing isometries. Isometries of three dimensional Euclidean space are products of at most three reflections. We enumerate the isometries **O** of the cube group in terms of their action on the twelve edges. Label the twelve edges 1-4 on top face and 5-8 on corresponding bottom face edges, so that 9-12 are the side edges connecting these.

(1) Identity
(2) 9-reflections: 3 with planes midway between opposite faces; 6 with planes passing through opposite edges. The cycle structures for action on the edges for the two types: e.g. of the type (15)(26)(37)(48)(9)(10)(11)(12) contributing $3x_1^4x_2^4$ to the cycle enumerator; of the type e.g. (9)(11)(10 12)(14)(23)(58)(67) contributing $6x_1^2x_2^5$
(3) 23-non-identity rotations-products of two reflections: either 8 rotations through opposite faces of order 4, contributing $8x_4^3$; 3 rotations of order 2 through opposite faces contributing $3x_2^6$; 8 rotations of order 3 through opposite vertices contributing $8x_3^4$.
(4) 15-compositions of rotation with reflection where the axis and the reflection plane are incident at a point: The reflection taking top face to bottom face is $\rho = (15)(26)(37)(48)(9)(10)(11)(12)$. Of the 23 non-identity rotations remove 8 with their axis lying in the reflection plane of ρ (2 opposite edges + 6 rotations in two opposite faces) leaving 15 rotations. Compose ρ with the 15 different rotations–8 (opposite vertices order 3), 2 (opposite faces order 4), 1 (opposite faces order 2), 4 (opposite edges) to get 15 different symmetries which are not reflections or rotations, called roto-reflections.

The cycle index enumerator is

$$P_{\mathbf{O}} = \frac{1}{48}(x_1^{12}+(3x_1^4x_2^4+6x_1^2x_2^5)+(6x_4^3+3x_2^6+6x_1^2x_2^5+8x_3^4)+(8x_6^2+2x_4^3+x_2^6+4x_4^3))$$
$$= \frac{1}{48}(x_1^{12}+3x_1^4x_2^4+12x_1^2x_2^5+12x_4^3+4x_2^6+8x_3^4+8x_6^2)$$

The enumerator for 2-colorings is

$$B^{12}+B^{11}W+4B^{10}W^2+9B^9W^3+18B^8W^4+24B^7W^5+$$
$$30B^6W^6+24B^5W^7+18B^4W^8+9B^3W^9+4B^2W^{10}+BW^{11}+W^{12};$$

there are 144 different 2-colorings and 30 of them have 6 colors each; there are 76 colorings which are distinct under switches of colors. We find that there are 8 color symmetric 2-colorings which are distinct using **O** [**1**, 5.38].

3. Recap

Using $\mathbf{O}^+$ there are 19 colorings and another 19 obtained by switches of colors and then 10 color symmetric ones to give a total of 48. On the other hand using the group **O** there are only 30 different 2-colorings: 11 colorings and their color switches together with the 8 symmetric colorings.

Hence to make different colorings we make 6 symmetric ones and label the other 2 pairs which have 'left-right' forms; for the non-symmetric case we make the 11 colorings which are rotation-reflection distinct and label the other 4 pairs of 'left-right' forms. This gives 23 'special' cubes.

From these 23 special cubes we can easily make the 48 cubes by switching colors or using a reflection: $2(11+2\cdot 4)+(6+2\cdot 2)=48$.

References

[1] N. G. De Bruijn, *Polya's Theory of Counting*, Chapter 5 in **Applied Combinatorial Mathematics**, E. Beckenbach editor, Wiley, 1964.

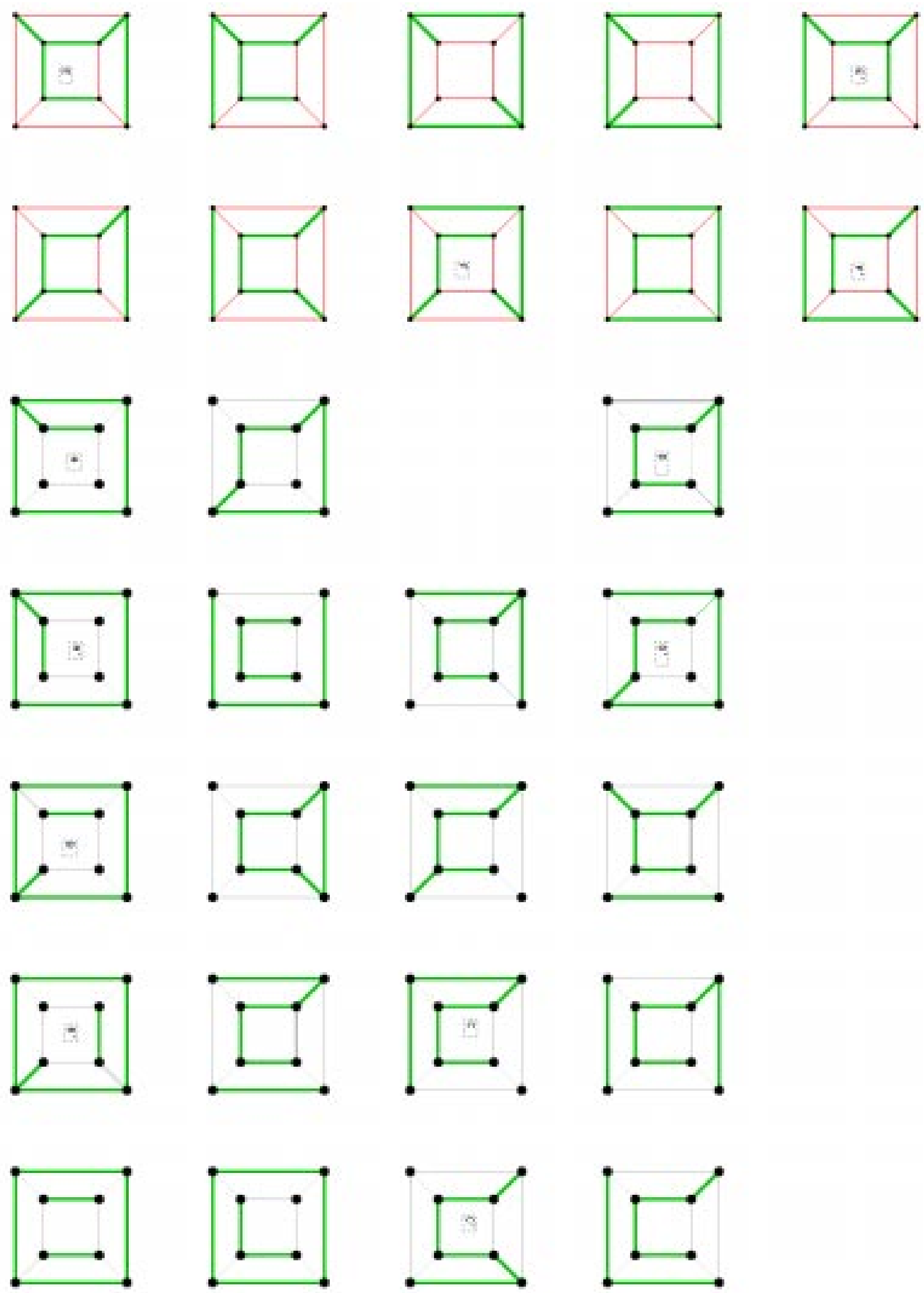

FIGURE 1. 29 2-Colorings