SOLVABLE GROUPS OF EXPONENTIAL GROWTH AND HNN EXTENSIONS

Roger C. Alperin †

An extraordinary theorem of Gromov, [Gv], characterizes the finitely generated groups of polynomial growth; a group has polynomial growth iff it is nilpotent by finite. This theorem went a long way from its roots in the class of discrete subgroups of solvable Lie groups. Wolf, [W], proved that a polycyclic group of polynomial growth is nilpotent by finite. This theorem is primarily about linear groups and another proof by Tits appears as an appendix to Gromov's paper. In fact if G is torsion free polycyclic and not nilpotent then Rosenblatt, [R], constructs a free abelian by cyclic group in G, in which the automorphism is expanding and thereby constructs a free semigroup. The converse of this, that a finitely generated nilpotent by finite group is of polynomial growth is relatively easy; but in fact one can also use the nilpotent length to estimate the degree of polynomial growth as shown by Guivarc'h, [Gh], Bass, [Bs], and Wolf, [W]. The theorem of Milnor, [M], on the other hand shows that a finitely generated solvable group, not of exponential growth, is polycyclic. Rosenblatt's version of this, [Rt], is that a finitely generated solvable group without a two generator free subsemigroup is polycyclic. We give another version of Milnor's theorem using the HNN construction. A consequence is that a finitely generated solvable group G has the ERF (extended residually finite) property iff G is polycyclic.

We briefly review the HNN construction. Generally, the HNN has a given base group B and two subgroups, H_1, H_2 together with an (external) element t of infinite order which conjugates H_1 to H_2, $\Gamma =< B, t \mid tH_1t^{-1} = H_2 >$. For solvable groups, a good example, is the group $\Gamma_1 =< a, t \mid tat^{-1} = t^2 >$. Many one relator groups have HNN decompositions; for example, consider $\Gamma_2 =< a, t \mid a = [tat^{-1}, t^2at^{-2}] >$. This is, in fact, the HNN extension with base $H =< a_0, a_1, a_2 \mid a_0 = [a_1, a_2] >$ and free subgroups $F_1 =< a_0 = a, a_1 = tat^{-1} >$, $F_2 =< tat^{-1}, a_2 = t^2at^{-2} >$ amalgamated, so that $\Gamma_2 =< H, t \mid tF_1t^{-1} = F_2 >$.

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These HNN constructions are ascending in the sense that a conjugation of the subgroup ascends or gets strictly larger in Γ. We say it is ascending with base B if Γ is generated by B and t, so that

$$\Gamma = < B, t \mid tBt^{-1} \subset B > .$$

In the first example, the cyclic group generated by a will by repeated conjugation by t^{-1} ascend to a group isomorphic to $Z[1/2]$; the square root of a exists since $(t^{-1}at)^2 = a$. In the second example above, the subgroup F_1 does not contain a_2, but F_2 does contain a_0 so this is properly ascending with base F_2 and conjugation by t^{-1}. Since the normal subgroup generated by a is perfect, and that subgroup is locally a free group it is infinitely generated. Brown, [Bn], considers such ascending 1-relator groups, and more general groups, in the context of actions on trees and HNN valuations.

It is well-known that a free product with amalgamation $A \underset{C}{*} B$ with C not of index less than or equal to 2 in each factor A, B, must contain a free semigroup (and even a free group). Without loss of generality choose $x \in A - C$, $y, z \in B - C$ distinct coset representatives, then xy, xz generate a free semigroup from the alternating word property for elements in free products with amalgamations. We do a similar construction for ascending HNN extensions, where the base is properly included, to obtain the following.

Lemma. *A properly ascending HNN extension with base B, $< B, t \mid tBt^{-1} \subset B >$, contains a two generator free semigroup.*

Proof. Let $B_i = t^iBt^{-i}$, $i \in Z$ and $T = \{t^n \mid n > 0\}$, the positive powers of t. For the proof of this lemma, we shall choose a coset representative $u \in B - B_1$, since the base is properly ascending. Moreover, B_1 is normalized by t, but not t^{-1}. Consider now the subsets $C = B_1 - \{1\}$, and $X = CT$. Because of the homomorphism $G \longrightarrow Z =< t >$, with kernel K, the normal subgroup generated by B, the exponent on t is well defined when an element of G is expressed as $K < t >$. Also the subgroup K is a properly ascending union of subgroups, $B_i \subset B_{i-1}$, for $i \in Z$.

The translate tX is a subset of X, since for $c \in C$, then $tct^{-1} \in C$ so that $t(ct^i) = (tct^{-1})t^{i+1}$. Consider also tuX; the element $tuct^i = (tut^{-1})(tct^{-1})t^{i+1}$ is contained in X; moreover since the exponent on t is positive and G is properly ascending, for $u \in B - B_1$, the element $(tut^{-1})(tct^{-1})$ is not 1. Finally, tX and tuX are disjoint subsets since if $tu(ct^i) = t(dt^j)$ $c, d \in C$ then $uc = d$, which is impossible by the choice of u.

Now from the disjointness of $X_1 = tX$ and $X_2 = tuX$, and also $X_1 \cup X_2 \subset X$, it now easily follows that t, tu generate a free semigroup. Any distinct words w_1, w_2 in t, tu without loss of generality begin on the left with t, tu respectively, so $w_1 X \subset X_1, w_2 X \subset X_2$ are different.

Since we can add free semigroup generators to a generating set the following is immediate.

Proposition. *A group which contains a free semigroup on two generators has exponential growth.*

Our version of Milnor's Theorem is the following.

Theorem A. *A finitely generated solvable group which is not polycyclic contains a subgroup which is a properly ascending HNN extension.*

Proof. Suppose that Γ is a finitely generated solvable group. Descending down the solvable series we obtain finitely generated layers up to level n, say, and infinitely generated mod Γ^{n+2} at $n+1$ or else the group is polycyclic. Consider the finitely generated solvable group $G = \Gamma^n/\Gamma^{n+2}$; $P = \Gamma^n/\Gamma^{n+1}$ is finitely generated abelian, and $A = \Gamma^{n+1}/\Gamma^{n+2}$ is infinitely generated abelian. Also P is not finite, since otherwise A is finitely generated. We may assume that P is torsion free by passing to a subgroup of finite index in G. It suffices now to prove that G contains a properly ascending HNN extension.

We claim that there is a $t \in G$ of infinite order, and subgroup $B \subset A$, so that $tBt^{-1} \subset B$ is proper. Let t be an arbitrary element of G of infinite order which maps non-trivially to P. Let $R = Z[t]$ be the group ring of the monoid generated by t. Let a be an arbitrary element of A; and let $M = Ra \subset A$ be the cyclic module generated by a. Certainly using 'module' action, $tM \subset M$; if this inclusion is proper we are done, $B = M$. So we may assume now that for every element t of P, $tM = M$. It follows that M is finitely generated as an abelian group, since there must be some polynomial which annihilates a. Thus using t and t^{-1} as above we have that for any element a, the set of all conjugates $t^i a t^{-i}, i \in Z$ is a finitely generated abelian subgroup of A.

Notice that since P is finitely presented, and G is finitely generated, then A is finitely generated as a normal subgroup of G. Let $\{a_1, a_2, ..., a_m\}$ be the finite set of normal generators of A; now using the finite set of generators $\{t_1, t_2, ..., t_n\}$ of G mod A, we obtain a finite set of generators for A. Consider $\{t_1^k a_i t_1^{-k} : i = 1, ...m, k \in Z\}$, since we have already shown that for any $a \in A$, $t^i a t^{-i}, i \in Z$ is finitely generated, this set is generated by a finite set of elements. Now continue with t_2 up to t_n. This gives a finite set of generators for A which contradicts our assumption on A; thus there must be a properly ascending HNN extension as a subgroup of G.

We extract the following Burnsidesque local-global property of a group Γ:
Property π: Given any finitely generated module A for Γ; if for every $t \in \Gamma$, $a \in A$, the $Z[t]$ module generated by a is finitely generated then A is a finitely generated abelian group.

Polycyclic groups have property π. If Γ is polycyclic we can use the normal form of elements, given from the polycyclic decomposition, to build iteratively as above, a finite set of generators starting from the finite set of module generators. In a similar way, we see that any group which has the following *bounded generation* property, has property π: there are a finite set of elements F, and a fixed integer N, so that every element of Γ has an expression as $g_1^{m_1} g_2^{m_2} ... g_N^{m_N}$, $g_i \in F$, $m_i \geq 1$. Many lattices in semisimple Lie groups have this bounded generation property; it is related to a positive solution to the congruence subgroup problem, [Rk]. Also, it is immediate, that a finitely presented torsion group with property π is finite. At the conference, J. S. Wilson kindly pointed out to me the following references. Kropholler [K] has shown that finitely generated minimax solvable groups have bounded generation. This includes the Baumslag-Solitar group Γ_1 discussed earlier. Brookes [Br], has investigated Engel elements of solvable groups and its relation to property π.

The following is immediate from the proof and discussion given above.

Theorem B. *A finitely generated group G containing a normal subgroup N, with infinitely generated abelianization N/N', contains a properly ascending HNN extension if G/N is a finitely presented group having property π.*

In particular this applies to $G = F/R'$ for an infinite group with finite (free) presentation F/R having property π. One can also replace R' by any other term $R^{(n)}, n \geq 2$ in the derived series of R to obtain similar results.

Recall that the group G has the property ERF if every subgroup is closed in the profinite topology, or equivalently, given any subgroup S and any element $x \in G - S$, there is a finite index subgroup (equivalently normal subgroup) containing S and not x; this is also the same as the existence of a finite quotient of G so that S is represented trivially and x non-trivially. The property LERF requires that only finitely generated subgroups are closed in the profinite topology. It has been shown by Malcev, [Mv], that a group which is a split extension, with normal subgroup having the ERF property, and quotient which has LERF, itself is ERF. This shows a large class of solvable groups have the property ERF. Since polycyclic groups satisfy the maximal condition on subgroups, all subgroups are finitely generated, we see immediately that a polycyclic group has the property ERF.

One can also apply our main Theorem A to obtain the following surprising result.

Theorem C. *A finitely generated solvable group G has the property ERF iff G is polycyclic.*

Proof. As remarked above, polycyclic groups have the property ERF. On the other hand, if the group is not polycyclic, we can find a subgroup B and element t so that $tBt^{-1} \subset B$, and $B_1 = tBt^{-1} \neq B$. Therefore in a finite image an element $a \in B - B_1$ can not be separated from B_1.

In particular, a polycyclic group is never a properly ascending HNN extension.

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alperin@mathcs.sjsu.edu
Department of Mathematics and Computer Science
San Jose State University, San Jose, CA 95192