Golomb Rulers

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INTRODUCTION

Consider a very simple 12 inch ruler with 13 equally spaced marks, 0 through 12, one inch apart. It is clear that using this ruler one can measure every one of the 12 integral lengths from 1 to 12 inches. However we don't need all of these 13 marks. For example, if we omit the mark at 2 then we can still measure the distance of length 2 since it is the distance between 1 and 3. In fact, there are lots of other ways we can measure the length 2.

Next, consider a ruler of length 6 with marks 0, 1, 4, and 6. An interesting property of this ruler is that each of the lengths 1, 2, 3, 4, 5 and 6 can be measured; in fact for each of these lengths it can be done in only one way.

The question arises whether it is possible to select a few marks on the ruler of length 12 with the same property: *Can every integral distance from 1 to 12 can be measured in only one way*? We will prove below (Theorem 1) that this is not possible.

Sets of integers (marks) having the property that if a distance can be measured using these marks then it can be done in a unique way are called Golomb rulers.

Definition 1. A set $\mathcal{G}$ of integers

$$a_1 < a_2 < \cdots < a_{p-1} < a_p$$

is called a *Golomb ruler* if for every two distinct pairs of these integers, say $a_i < a_j$ and $a_m < a_n$, we have $a_j - a_i \neq a_n - a_m$.

The *size* of $\mathcal{G}$ is defined to be p (the number of marks in $\mathcal{G}$) and is denoted $\#\mathcal{G}$. The *length* of $\mathcal{G}$ is defined to be $a_p - a_1$ (the largest distance that can be measured using the marks from $\mathcal{G}$).

It is clear that if $\mathcal{G} = \{a_1, a_2, \cdots, a_p\}$ is a Golomb ruler then so is $\{\mathrm{a}_1 + b, a_2 + b, \cdots, a_p + b\}$, i.e., the choice of a_1 is immaterial and will usually be taken to be 0. It is also clear that if $\mathcal{G} = \{a_1, a_2, \cdots, a_p\}$ is a Golomb ruler, then so is the reflection of $\mathcal{G}$ around the midpoint $(a_1 + a_p)/2$. For example, $\{0, 1, 4, 6\}$ is a Golomb ruler, as is $\{0, 2, 5, 6\}$ obtained by reflecting the first ruler around the point 3.

For convenience, a set $\{a_1\}$ consisting of a single point is considered to be a Golomb ruler. This simplifies the statements of some of the theorems. It is clear that given a Golomb ruler $\mathcal{G}$ of size p and length N, the value of a_1 can be chosen arbitrarily and that $a_p = a_1 + N$.

Golomb rulers have numerous applications. The best known is an application to radio astronomy. Radio telescopes (antennas) are placed in a linear array. For each pair of these antennas, the received signals are subtracted from each other and an inference can be then made as to the location of the source. These inferences can be made much more accurate if all the distances between the antennas are multiples of the same common length, and many such pairs with distinct distances between them are available and can be utilized. The problem maximizing the number of distinct distances between the pairs, while minimizing the number of the antennas and the length of the array, was first considered by Solomon W. Golomb in [8]. (See also J. Blum, F. Biraud, and J. C. Ribes [1], G. S. Bloom and S. W. Golomb [2] and A. T. Moffet [10]). Other applications include assignments of channels in radio communications, X-ray crystallography, and self-orthogonal codes.

See W. Rankin [12] for more information on these applications. There is also a wealth of information in various writings by Martin Gardner [5], [6] [7].

The example of the Golomb ruler $\{0, 1, 4, 6\}$ was given above. This ruler has the additional property that every integral distance between 1 and 6 can be measured. We call such a ruler perfect.

DEFINITION 2. A Golomb ruler $\mathcal{G}$ of length N is called *perfect* if every integer d, $1 \leq d \leq N$, can be expressed as $d = a - a'$, for some $a, a' \in \mathcal{G}$. (Since $\mathcal{G}$ is a Golomb ruler, this representation of d is unique.)

Unfortunately, there are very few perfect Golomb rulers.

THEOREM 1. (Golomb) *Together with their translations and reflections around the midpoint the only perfect Golomb rulers are* $\{0\}, \{0, 1\}$*,* $\{0, 1, 3\}$*, and* $\{0, 1, 4, 6\}$.

This theorem was proved by Golomb, but apparently he never published it. There are several places where the proof appears (A. Dimitromanolakis [4] or W. Rankin [12]), but they are not very easily accessible, so we present here a slight modification of the original argument.

Proof. If $\mathcal{G}$ is a perfect Golomb ruler of size p and length N, we must have $N = \binom{p}{2} = \frac{1}{2}p(p-1)$. This is clear enough. Given p points, the number of distinct pairs of these points is $\binom{p}{2}$. Since the length of $\mathcal{G}$ is N, there are N distances to be measured, so $N = \binom{p}{2}$, i.e., N must be a triangular number. The triangular numbers below 10 are $0, 1, 3, 6$ and the corresponding rulers are listed in the theorem

Assume then that $\mathcal{G}$ is a perfect Golomb ruler of length $N > 9$ and we seek a contradiction. Without loss of generality we may assume that a_1, the smallest number in $\mathcal{G}$, is equal to 0 and so the largest number is $a_p = N$. By hypothesis, every number $1 \leq d \leq N$ is uniquely realizable as a difference of two marks in $\mathcal{G}$. Since N is realizable, 0 and N must belong to $\mathcal{G}$. Since $N - 1$ is realizable, either 1 or $N - 1$ belongs to $\mathcal{G}$. By reflecting $\mathcal{G}$ around $N/2$, we may assume that $1 \in \mathcal{G}$. Next, since $N > 3$ then $N - 2$ must be realized. Since $N - 2 > 1$ then $\mathcal{G}$ must contain another point.

The possible pairs realizing $N - 2$ are:

$\{2, N\}$, which implies $2 \in \mathcal{G}$ and produces a duplication: $1-0=2-1$

$\{1, N-1\}$ which implies $N-1 \in \mathcal{G}$ and a duplication: $1-0=N-(N-1)$

$\{0, N-2\}$ which is okay, i.e., $\mathcal{G}$ must contain the point $N-2$, in addition to the points 0, 1, and N.

Thus $0, 1, N-2$ and $N \in \mathcal{G}$ and the realized distances are $1, 2, N-3, N-2, N-1, N$.

Next, $N-4$ has to be realized, it has not yet realized since $N-4 \notin \{1, 2\}$. (Have patience, the process stops when we consider $N-5$.)

The possible pairs for realizing $N-4$ are:

$\{0, N-4\}$, so that $N-4 \in \mathcal{G}$ producing the duplication $(N-2)-(N-4)=N-(N-2)$;

$\{1, N-3\}$, so that $N-3 \in \mathcal{G}$ producing the duplication $1-0=(N-2)-(N-3)$;

$\{2, N-2\}$, so that $2 \in \mathcal{G}$ producing the duplication $2-0=N-(N-2)$;

$\{3,\ N-1\}$, so that $N-1 \in \mathcal{G}$ producing the duplication $1-0=N-(N-1)$;

$\{4, N\}$, which is okay, i.e., $\mathcal{G}$ contains the point 4.

At this point, $\mathcal{G}$ is forced to have the following numbers: 0, 1, 4, $N-2$, and N. Hence $1, 2, 3, 4, N-6, N-4, N-3, N-2, N-1$ and N are distances realized in $\mathcal{G}$.

Finally, consider the distance $N-5$. It is not yet realized, since $N-5 \notin \{1, 2, 3, 4\}$. (Here we need $N > 9$.)

The possible pairs for realizing the distance $N-5$ are:

$\{0, N-5\}$, so that $N-5 \in \mathcal{G}$, which produces the duplication $(N-2)-(N-5)=4-1$;

$\{1, N-4\}$, so that $N-4 \in \mathcal{G}$, which produces the duplication $(N-2)-(N-4)=N-(N-2)$;

$\{2, N-3\}$, so that $2 \in \mathcal{G}$, which produces the duplication $1-0=2-1$;

$\{3, N-2\}$, so that $3 \in \mathcal{G}$, which produces the duplication

$3 - 0 = 4 - 1$;

$\{4, N-1\}$, so that $N - 1 \in \mathcal{G}$, which produces the duplication $1 - 0 = N - (N - 1)$;

$\{5, N\}$, so that $5 \in \mathcal{G}$, which produces the duplication $1 - 0 = 5 - 4$. These duplications give a contradiction and this completes the proof.

Since perfect Golomb rulers essentially do not exist, we seek "almost perfect" rulers. Roughly speaking, given a length N, we try to place as many points as possible in the interval $[0, N]$ so that the resulting set forms a Golomb ruler. Alternatively, given the size p of the ruler (the number of marks), we try to construct a Golomb ruler of shortest possible length N with p points. Such rulers are called optimal.

Definition 3. For every positive integer p, let $G(p)$ be the shortest possible length of a Golomb ruler with p marks.

A Golomb ruler with p marks is called *optimal* if its length is $G(p)$.

For example, $G(6) = 17$, and there are 4 optimal rulers of size 6 and length 17:

$$\{0, 1, 4, 10, 12, 17\}, \{0, 1, 4, 10, 15, 17\}, \{0, 1, 8, 11, 13, 17\}, \{0, 1, 8, 12, 14, 17\}.$$

Optimal rulers have been found by a computer search, and for larger values of p these searches are extremely time consuming. The largest known value of $G(p)$ is, at the present time, $G(26) = 492$ and the corresponding optimal Golomb ruler has marks

$$0\ 1\ 33\ 83\ 104\ 110\ 124\ 163\ 185\ 200\ 203\ 249\ 251\ 258$$

$$314\ 318\ 343\ 356\ 386\ 430\ 440\ 456\ 464\ 475\ 487\ 492$$

It is not known if it is unique and the search took several years (J. W. Shearer [13] and E. W. Weisstein [14]). Wikipedia is a also good source of information on the latest status of the values of $G(p)$.

Given a Golomb ruler with p marks, there are $\binom{p}{2} \sim \frac{1}{2}p^2$ distinct distances one can measure with this ruler. Thus, one expects $G(p)$ to be roughly at least $\frac{1}{2}p^2$. It is a conjecture, with strong empirical evidence, that $G(p) < p^2$; but it is only a conjecture.

The concept of a Golomb ruler has also a close connection with additive number theory. It is completely outside the scope of this paper to discuss this connection in any depth, and we only state some facts and invite the reader to investigate further.

Definition 4. A subset $\mathcal{B}$ of integers contained in $[1, N]$ is called a B_2 *basis*, if for any two distinct pairs of integers from $\mathcal{B}$, say a, a' and b, b' we have

$$a + a' \neq b + b'.$$

It is clear that if $\mathcal{G}$ is a B_2 basis then $\mathcal{G}$ is a Golomb ruler, and conversely. There is an old conjecture of Erdös and Turan which states that a B_2 basis with $\lfloor \sqrt{N} \rfloor$ elements can be constructed in $[1, N]$ for any N. This is very closely related to the conjecture that $G(p) < p^2$. For a comprehensive discussion of the additive number theory see H. Halberstam and F. K. Roth [9] and for the connection between these two concepts, as well as for a detailed discussion of optimal Golomb rulers, see A. Dimitromanolakis [4].

Perfect Rulers on $\mathbb{N}$

In this section we study *infinite rulers*. These are sets $\mathcal{G}$ of non-negative integers, such any positive integer d is realized as a distance between some two elements of $\mathcal{G}$. In addition, if we require that this representation is unique, we may speak of infinite perfect Golomb rulers.

Definition 5. A subset $\mathcal{G}$ of the set $\mathbb{N}$ of natural numbers is called an *infinite perfect Golomb ruler* if

1) For every positive integer d, there are elements $a, a' \in \mathcal{G}$ so that $d = a - a'$
2) For every such d this representation is unique.

It is not entirely clear that such things exist, but in fact they do and also they can be made arbitrarily *thin* (sparse) depending on the choice of a function φ.

Theorem 2. *Let $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be strictly increasing with $\varphi(x) \to \infty$ as $x \to \infty$. Then there is a subset $\mathcal{G}$ of $\mathbb{N}$ which is an infinite perfect Golomb ruler and such that for $x > x_0 = x_0(\mathcal{G}, \varphi)$*

$$\#\{k \mid k \in \mathcal{G},\ k \le x\} \le \varphi(x). \tag{1}$$

Proof. The basic idea for construction of the set $\mathcal{G}$ is as follows: We choose a rapidly increasing sequence $\gamma_k, k = 1, 2, \cdots$, and construct $\mathcal{G}$ by successively adding the points $\{\gamma_k, \gamma_k + k\}$. If a duplication should occur as a result of this addition then we do not add the pair. Various things have to be proved, for example, that skipping a pair does not result in some integer d not being realized as the difference of two element of $\mathcal{G}$, etc. The details follow.

Choose a strictly increasing function $\psi(x)$such that

$$x < \frac{1}{2}\varphi(\psi(x)) \tag{2}$$

and define a sequence $\{\gamma_k\}$ by

$$\begin{aligned} &\gamma_1 = 0 \\ &\gamma_{k+1} > \psi(k+1) + 2(\gamma_k + k) + 1, \text{ for } k \ge 1. \end{aligned} \tag{3}$$

Define now $\mathcal{A}_1 = \{\gamma_1, \gamma_1 + 1\}$ which of course equals $\{0, 1\}$ and

$$\mathcal{A}_{k+1} = \begin{cases} \mathcal{A}_k \cup \{\gamma_{k+1}, \gamma_{k+1} + (k+1)\} \\ \qquad \text{if this set } \mathcal{A}_k \cup \{\gamma_{k+1}, \gamma_{k+1} + (k+1)\} \text{ has no duplicate distances} \\ \mathcal{A}_k \text{ otherwise.} \end{cases}$$

Basically, start with $\{\gamma_1, \gamma_1 + 1\}$ and then for each $k > 1$ we add two new points $\{\gamma_k, \gamma_k + k\}$ provided that this does not introduce a duplication. Finally, we set

$$\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{A}_k.$$

First of all we show that the set $\mathcal{G}$ satisfies the density condition (1) in the statement of the Theorem. Let $x > 1$ be given and let k_0 be the largest integer such that $\gamma_{k_0} \leq x$. Then clearly

$$\#\{k \mid k \in \mathcal{G}, k \leq x\} \leq 2k_0$$

because the elements of $\mathcal{G}$ come in pairs: γ_p and $\gamma_p + p$. Now

$$k_0 < \frac{1}{2}\varphi(\psi(k_0)) < \frac{1}{2}\varphi(\gamma_{k_0}) \leq \frac{1}{2}\varphi(x).$$

The first inequality follows from (2) and the second and third follow from (3) and the fact that φ is monotonically increasing. Thus the density claim (1) of the Theorem is true.

By construction, there are is no duplication of distances in $\mathcal{G}$. This is quite clear, since we made sure that there is no duplication of distances in any of the sets $\mathcal{A}_k$.

What remains to be shown is that every distance d is realized as a difference of two elements of $\mathcal{G}$. Clearly it suffices to show the following: Suppose that during the examination of a pair $\{\gamma_p, \gamma_p + p\}$ for an inclusion in the set $\mathcal{A}_p$ it is determined that a duplication would occur if that pair is included. Then

$$p = a - a' \text{ where } a, a' \in \mathcal{A}_{p-1} \tag{4}$$

i.e., p is already realized as a distance in the set $\mathcal{A}_{p-1}$. This would occur, for example, in the following situation: Let a and a' be two points in a set $\mathcal{A}_q$, for some q such that $a - a' > q$. Then, further along in the process, adding the pair $\{\gamma_{a-a'}, \gamma_{a-a'} + (a - a')\}$ would surely create a duplication. The claim is that this is essentially the only way it could happen. Now, if this claim is true, then either every distance d occurs in $\mathcal{G}$ through the addition of the pair $\{\gamma_d, \gamma_d + d\}$ or d occurs already as a distance in the set $\mathcal{A}_{d-1}$.

We now prove the assertion (4). Suppose that the addition of the pair $\{\gamma_p, \gamma_p + p\}$ to the set $\mathcal{A}_{p-1}$ results in duplications. Because there are no duplications in the set $\mathcal{A}_{p-1}$ these duplications must involve the points from the pair under discussion. It follows, because of (3), that both points of the pair are larger than any of the points in $\mathcal{A}_{p-1}$, and so the possibilities are:

$$
\begin{aligned}
&i)\ (\gamma_p + p) - \gamma_p = a - a' \\
&ii)\ (\gamma_p + p) - a = \gamma_p - a' \\
&iii)\ (\gamma_p + p) - a = a' - a'' \\
&iv)\ \gamma_p - a = a' - a''
\end{aligned}
$$

where a's are elements of the set $\mathcal{A}_{p-1}$. In cases $i)$ and $ii)$ then p is a difference of some elements in $\mathcal{A}_{p-1}$, i.e. (4) holds. The possibilities $iii)$ and $iv)$ cannot occur because the largest element of $\mathcal{A}_{p-1}$ is at most $\gamma_{p-1} + (p-1)$ and from (3) then $\gamma_p > 2(\gamma_{p-1} + p - 1)$. But, if either $iii)$ or $iv)$ were true, then either γ_p or $\gamma_p + p$ would be at most twice the largest element of $\mathcal{A}_{p-1}$. Thus our claim (4) is shown and the theorem is proved.

Thus, thin infinite perfect Golomb rulers exist. The procedure of constructing these given in Theorem 2 does not give a "formula" for the n^{th} mark— it just constructs these marks one by one.

It would be interesting to know how *thick* an infinite perfect Golomb can be. In particular is it possible to have

$$\delta_{\mathcal{G}}(x) = \#\{\ k \mid k \in \mathcal{G}, k \le x\} \sim \sqrt{x}? \tag{5}$$

By arguments similar to the discussion of finite perfect Golomb rulers, it is easy to see that $\delta_{\mathcal{G}}(x)$ should roughly by at least $\frac{1}{2}x^2$, and (5) is motivated by the Erdös-Turan conjecture (it does not follow from nor does it imply the conjecture). Unfortunately we are unable to settle the issue raised by the question in (5).

Minimal Spanning Rulers

In this section we consider a related concept of rulers of finite length, which can be used to measure every distance. They differ from Golomb rulers in that there might be a distance which can be measured in two different ways, but we require that every eligible distance can be measured. We call such rulers spanning.

Definition 6. Let $\mathcal{S} = \{0 = a_1 < a_2 < \cdots < a_p = N\}$ be a set of integers. We say that $\mathcal{S}$ is a *spanning ruler on* $[0, N]$ if every integer $1 \le d \le N$ can be expressed as $d = a - a'$, with a and $a' \in \mathcal{S}$.

We say that a spanning ruler $\mathcal{M}$ is *minimal* on $[0, N]$, if whenever $\mathcal{M}'$ is a proper subset of $\mathcal{M}$ then the set $\mathcal{M}'$ is not a spanning ruler on $[0, N]$.

Minimal spanning rulers obviously exist. Just start with $\{0, 1, \cdots, N\}$ and remove one point at a time until you can't do it anymore.

However, minimal rulers cannot be very "thin". If $\mathcal{M}$ is a minimal ruler of length N and $p = \#\mathcal{M}$, then $\binom{p}{2} = \frac{1}{2}p(p-1) \ge N$; so p is roughly at least $\sqrt{2N}$. We now show that we can come fairly close to this lower bound.

Theorem 3. *For every integer $N \ge 4$ there is a minimal spanning ruler $\mathcal{M}_N \subset [0, N]$ such that*

$$2\sqrt{N} - 1 \le \#\mathcal{M}_N < 2\sqrt{N} \tag{6}$$

and the equality on the left side holds only when N is a perfect square.

Proof. The basic idea of the proof can best be seen by an example of a thin minimal ruler for $N = 100$. The ruler $\mathcal{M}_{100}$ is in this case taken to be

$$\mathcal{M}_{100} = \{0, 1, 2, 3, \cdots, 9, 20, 30, 40, \cdots, 90, 100\}.$$

Notice that the number *10 is not* included. The number of elements in $\mathcal{M}_{100}$ is 19 which is equal to $2\sqrt{100} - 1$. Every distance $1 \le d \le 100$ is realizable: for $d = 10 = 30 - 20$ say whereas any other multiple d of 10 is $d = d - 0$. If $d = q \cdot 10 + j$, $1 \le q, j \le 9$ then $d = (q + 1) \cdot 10 - (10 - j)$.

Finally, if $1 \le d \le 9$ then $d = d - 0$. None of the numbers can be removed. For example, $d = 7$ cannot be removed because then $13 = 20 - 7$ would not be realizable. The number 30 cannot be removed because then $21 = 30 - 9$ would not be realizable. If 10 is included the ruler is not minimal. The actual proof is based on this example although some care has to be taken when N is not a perfect square. Here are the details.

By inspection, when $N \in \{5, 6, 7, 8\}$ then the minimal spanning rulers satisfying (6) are, respectively:

$$\{0, 1, 3, 5\}, \{0, 1, 4, 6\}, \{0, 1, 4, 5, 7\}, \{0, 1, 4, 6, 8\}.$$

Incidentally, there are no minimal spanning rulers satisfying the condition (6) for $N \in \{1, 2, 4\}$ and there is one for $N = 3$, namely $\{0, 1, 3\}$.

Let $\xi = \left\lfloor \sqrt{N} \right\rfloor$ so that $\xi^2 \leq N < (\xi+1)^2 = (\xi+2)\xi+1$. We assume that $\xi \geq 3$. There are two possibilities:

$$\alpha)\ N = 0 \bmod \xi \text{ so that } N = K\xi,\ K = \xi,\ \xi+1, \text{ or } \xi+2 \tag{7}$$

$$\beta)\ N \neq 0 \bmod \xi \text{ so that } N = K\xi+\eta,\ K = \xi \text{ or } \xi+1, \text{ and } 1 \leq \eta < \xi.$$

In case α) take

$$\mathcal{M}_N = \{0, 1, \cdots, \xi-1, 2\xi, 3\xi, \cdots, K\xi\} \quad (\xi \text{ is not included})$$

where K is as in (7).

Every distance $1 \leq d \leq N$ is realized as the following analysis shows:

If $1 \leq d \leq \xi - 1$, $d = d - 0$;

If $d = \xi$, $d = 3\xi - 2\xi$ since both $2\xi, 3\xi \in \mathcal{M}_N$ for $\xi \geq 3$;

If $d = q\xi, q > 1$, then $d = q\xi - 0$;

If $d = q\xi + \eta, 1 \leq q < K, 1 \leq \eta < \xi$, then $d = (q+1)\xi - (\xi - \eta)$.

Also, we see that none of the marks can be removed:

The endpoints 0 and N cannot be deleted because $N = N - 0$;

The points $1 \leq d < \xi$ cannot be deleted because of the distance $\xi + (\xi - d) = 2\xi - d$;

The points $q\xi, 2 \leq q \leq K$, cannot be deleted because of the distance $(q-1)\xi + 1 = q\xi - (\xi - 1)$.

In addition we have that $\#\mathcal{M}_N = \xi + K - 1$. Thus, to show (6) we must prove that for $t = 0, 1, 2$ then

$$2\sqrt{\xi(\xi+t)} - 1 \leq 2\xi + t - 1 < 2\sqrt{\xi(\xi+t)}$$

with equality holding on the left side only when $t = 0$. This is done in a straightforward matter by squaring each side of the inequality to eliminate $\sqrt{\cdots}$ expressions.

In case β) take

$$\mathcal{M}_N = \{0, 1, \cdots, \xi - 1, 2\xi, 3\xi, \cdots, K\xi, K\xi + \eta\} \quad (\xi \text{ not included})$$

where K, η are as specified in (7). Again, all the distances $1 \leq d \leq N = K\xi + \eta$ can be realized: If $1 \leq d \leq K\xi$ then the argument is the same as in case α); If $d = K\xi + \delta, 1 \leq \delta \leq \eta$, $d = (K\xi + \eta) - (\eta - \delta)$.

None of the marks can be removed:

The endpoints cannot be removed because of the distance $N = N - 0$.

The points $\xi - c$, $1 \leq c < \xi$, $c \neq \eta$ cannot be removed because of the distance $\xi + c = 2\xi - (\xi - c)$.

The point $\xi - \eta$ cannot be removed because of the distance

$$(K - 1)\xi + \eta = K\xi - (\xi - \eta). \tag{7}$$

However also $(K - 1)\xi + \eta = (K\xi + \eta) - \xi$ and we see that (7) is the only way to realize the distance $(K - 1)\xi + \eta$ since $\xi \notin \mathcal{M}_N$

The points $2\xi, 3\xi, \cdots, K\xi$ cannot be removed for the following reason: Let τ be such that $\tau \neq \eta$, $1 \leq \tau < \xi$. If $k\xi$ is removed then the distance $(k - 1)\xi + \tau = k\xi - (\xi - \tau)$ is not realizable. (It can't be realized using the mark $K\xi + \eta$.)

Finally, $\#\mathcal{M}_N = \xi + K$ and to show (6) we must prove that for $t = 0, 1$ and $1 \leq \eta < \xi$ then

$$2\sqrt{\xi(\xi + t) + \eta} - 1 < 2\xi + t < 2\sqrt{\xi(\xi + t) + \eta}.$$

Again, squaring both sides of each inequality to eliminate $\sqrt{\cdots}$ and doing some algebra does the job.

The minimal spanning rulers can also be quite *thickly* marked.

THEOREM 4. *For any $N > 0$ there is a minimal spanning ruler $\mathcal{M}_N$ with*

$$\#\mathcal{M}_N > \frac{1}{2}N.$$

Proof. A moment's reflection shows that if $N = 2n$ or $N = 2n + 1$ then $\mathcal{M}_N = \{0, 1, \cdots, n, N\}$ works.

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