Origami Alignments and Constructions in the Hyperbolic Plane

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1 Introduction

Neutral geometry is the geometry made possible with the first 28 theorems of Euclid's Book 1—those results that do not rely on the parallel postulate. Hyperbolic geometry diverges from Euclidean geometry in that there are two parallels (asymptotic) to a given line through a given point and infinitely many other lines through the point that do not meet the given line (ultraparallel). In hyperbolic geometry, similar triangles are congruent; triangles have less than 180 degrees; there are no squares (regular four sided with 90 degree angles); and the area of a triangle is its defect (radian difference between π and the angle sum). As a consequence of this last remark, Bolyai (circa 1830) showed that one can square some circles (one uses a regular four sided polygon) in the hyperbolic plane using a ruler and compass [Gray 04, Curtis 90, Jagy 95]. Our aim here is to introduce origami constructions as a new method for doing geometry in the hyperbolic plane. We discuss the use of origami for making any ruler-compass construction and the possibilities for other constructions that are not ruler-compass constructions.

In the first section, we give the alignments for folding in the hyperbolic plane. These are the analogues of the classical origami axioms in the plane discussed by Huzita and Justin. There is one additional alignment that is possible. Next, we discuss the relations between the alignments in the context of neutral geometry, Euclidean geometry, and hyperbolic geometry. This leads up to the the relation of these alignments to the ruler-compass constructions in the hyperbolic plane. Basically, we show that one can do ruler-compass constructions equivalently with a subset of the folds, similar to origami in the Euclidean plane.

We use the projective model (Cayley-Klein) for the hyperbolic plane [Fishback 69]. The model is the interior of the unit disk; the bounding circle is called the *absolute*. The points and lines of this geometry are the same as points and lines of the plane that lie in the the interior of the disk. Since the model embeds in the Euclidean plane, we can make these folds using some of the Euclidean origami folds—we call this a *simulation*. We also discuss the relations of ruler-compass methods to the coordinates of the constructed points.

It is important to use the projective model. We show in Section 5 that the non-Euclidean parabola involved in axioms $\mathbb{H}_5$ and $\mathbb{H}_6$ is a conic and thus by Bezout's theorem there are at most four common tangents to a pair of these curves.

Finally, we discuss the use of the fold $\mathbb{H}_6$, which accomplishes geometrical constructions that are not ruler-compass constructions. These allow constructions that can be used to solve cubic and quartic equations. We show that the real subfield of Euclidean origami numbers [Alperin 00] can be realized with these constructions. For this, we show how to construct cube roots and trisections of angles in hyperbolic geometry.

2 Basic Alignments and Folds

2.1 Alignments

A *fold line* is the axis of a perpendicular reflection. We want to find (minimal) *alignments* of points and lines that are brought into coincidence (aligned) by one reflection. A fold line is determined by specifying its two degrees of freedom. The basic alignments and partial alignments (using only one degree of freedom) are the following:

1. *Fold two points together:* $P \leftrightarrow Q$ (uses two degrees of freedom).

   If the points are the same, $P \leftrightarrow P$ means the point must be on the fold line. There is one degree of freedom remaining.

2. *Fold two lines together:* $L \leftrightarrow M$ (uses two degrees of freedom).

   If the lines are equal, $L \leftrightarrow L$ yields a fold perpendicular to the L; this uses one degree of freedom.

3. *Fold a point and a line together:* $P \leftrightarrow L$ (uses one degree of freedom).

For the fold when P is on L, there is just one degree used; the folds produced are either perpendicular to L or fold lines pass through P.

The partial alignments are now combined with other partial alignments to give an alignment or fold line. An alignment can thus be expressed using either one or two $\leftrightarrow$ symbols.

2.2 Folds

We consider the alignments that describe a line or finite set of lines in the hyperbolic plane using the projective model of Cayley-Klein as the interior of a circle (the absolute). The lines and points are ordinary lines that meet the interior of the absolute and ordinary points interior to the absolute.

We use the notation $\mathbb{H}$ to denote the alignments in the hyperbolic plane.

2.3 Unique Alignment Folds

The following alignment rules determine at most one line.

$\mathbb{H}_0$: $L \leftrightarrow L$, $M \leftrightarrow M$. This fold is the unique perpendicular to two ultraparallel lines. (See Figure 1.)

$\mathbb{H}_1$: $P \leftrightarrow P, Q \leftrightarrow Q$. This alignment or fold is the line passing through two points. (See Figure 2.)

$\mathbb{H}_2$: $P \leftrightarrow Q$. This fold is the perpendicular bisector of PQ. (See Figure 2.)

$\mathbb{H}_3$: $P \leftrightarrow P$, $L \leftrightarrow L$. This fold is the perpendicular to L through P. (See Figure 3.)

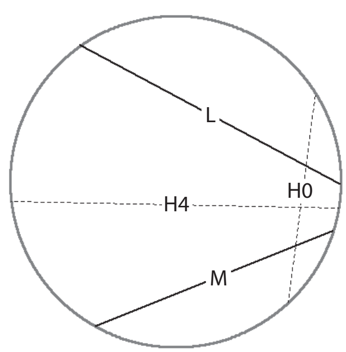

Figure 1. $\mathbb{H}_0$ is the common perpendicular, and $\mathbb{H}_4$ is the midline of L, M.

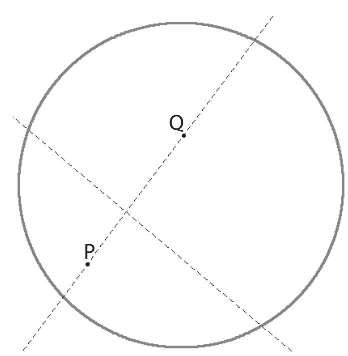

Figure 2. $\mathbb{H}_1$ is the line through P, Q, and $\mathbb{H}_2$ is the perpendicular bisector of P, Q.

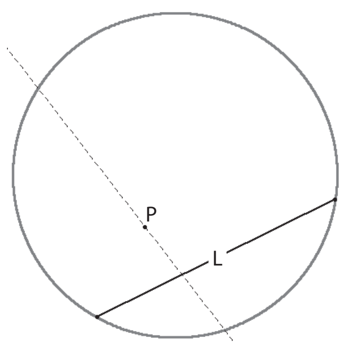

Figure 3. $\mathbb{H}_3$ is the perpendicular to L through P.

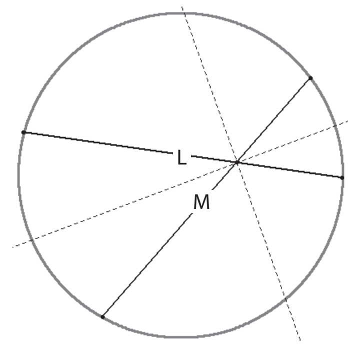

Figure 4. $\mathbb{H}_4$ gives the two perpendicular angle bisectors of L, M.

2.4 Quadratic Folds

The above alignments have two solutions in general.

$\mathbb{H}_4$: $L \leftrightarrow M$. These alignment folds are the two (perpendicular) angle bisectors. (See Figure 4.)

$\mathbb{H}_5$: $P \leftrightarrow L,\ Q \leftrightarrow Q$.

All the reflections of P in the pencil of lines at Q (using $Q \leftrightarrow Q$) form the circle centered at Q passing through P. The intersection points of this circle with L are the reflected images P_1, P_2 of P. The hyperbolic circle with center Q appears as an ellipse in the projective model, which passes through P with minor axis OQ (O is the center of the absolute). (See Figure 5.)

An alternative interpretation of these folds is that they are the tangents to a non-Euclidean parabola with focus P and directrix L, where the fold lines pass through Q. We discuss this in more detail in Section 5.

$\mathbb{H}_{5'}$: $P \leftrightarrow L,\ M \leftrightarrow M$.

The reflections of P in all the perpendiculars to M (using $M \leftrightarrow M$) gives the equidistant curve to M passing through P. The intersections of this equidistant curve with L gives at most two reflected images whose perpendicular bisectors with P are the fold lines. The line L may meet the equidistant curve in 0, 1, or 2 points on the same side of M as P. (See Figure 6.)

The equidistant curve appears in the projective model as a branch of a conic tangent to the absolute at the ends of M and passes through P.

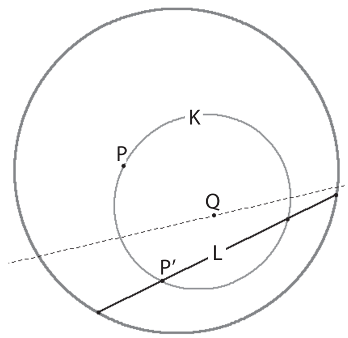

Figure 5. $\mathbb{H}_5$ with circle K having center Q and radius QP.

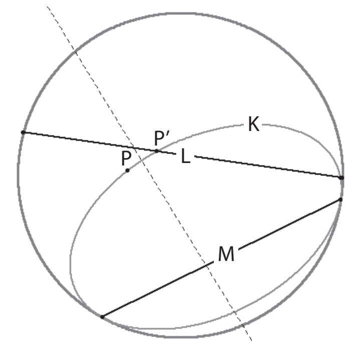

Figure 6. $\mathbb{H}_{5'}$ with equidistant curve K having axis M and passing through P.

2.5 Quartic Folds

These alignments determine four lines in general.

$\mathbb{H}_6$: $P \leftrightarrow L,\ Q \leftrightarrow M$.

The reflection of Q in the tangents to the non-Euclidean parabola, enveloped by the folding $P \leftrightarrow L$, give a quartic curve. It is a rational singular quartic related to the constructions of pedals of conics [Alperin 04]. One of the singularities of this quartic is at Q. Intersections of the line M with this quartic give four possible solutions. In the Euclidean case, one of the four common tangents is at infinity so there are at most three fold lines (see Figure 7). In the non-Euclidean case we may have some of these four possible fold lines outside of the absolute (see Figure 8).

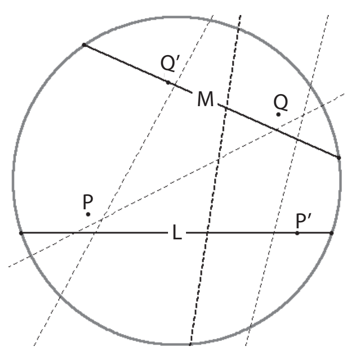

Figure 7. $\mathbb{H}_6$ with fold line P to P', Q to Q'.

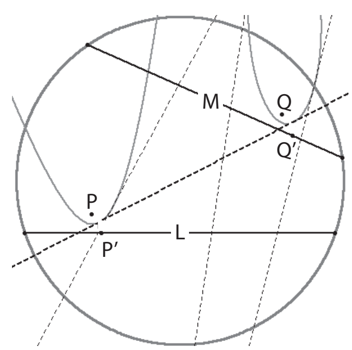

Figure 8. $\mathbb{H}_6$ with non-Euclidean parabolas shown.

3 Relations between the Alignment Axioms

We first investigate relations between these alignments in the context of neutral geometry. We use the notation O_i (for neutral geometry) rather than $\mathbb{H}_i$, indicating that we are not using any properties of parallels.

Theorem 1. *In neutral geometry with* O_1, $O_6 \to O_5$, $O_5 \to O_4$, $O_5 \to O_3$, $O_5 \to O_2$, $O_4 \to O_2$, *and* $O_6 \to O_{5'}$, $O_{5'} \to O_3$, $O_{5'} \to O_2$.

Proof: $O_6 \to O_5$: Let $M = PQ$. Folding Q onto M means the fold passes through Q or is perpendicular to M. The folds through Q are those which satisfy O_5.

$O_5 \to O_4$: Fold P onto $L = QR$ (passing through Q) gives the angle bisectors of $\angle PQR$. If the lines don't meet, then use Bolyai's construction: place two points P, P' on L and Q, Q' on M. Construct, using O_1, the line PQ and $P'Q'$. Construct angle bisects at P, Q meeting at R and P', Q' meeting at R'. The line RR' is the desired fold line.

$O_5 \to O_3$: Choose point Q on L (and different from P); now fold, using O_5, the line that reflects Q to L and passes through P. This line must be perpendicular to L and pass through P.

$O_5 \to O_2$: Make perpendiculars at P and Q on PQ; bisect these right angles using O_4; now use O_1 to connect intersections of bisectors giving the perpendicular bisector.

$O_4 \to O_2$: Fold P onto L. The bisection of the 180° angle at P is the same as the perpendicular to P at L. For two points P, Q we construct the perpendiculars at P, Q to the line $L = PQ$. Now bisect these right angles, and then construct the perpendicular bisector of PQ in the standard way using a line through the intersections of the corresponding bisectors.

$O_6 \to O_{5'}$: Fold P on L and Q on M when Q is on M; this is same as folding P to L and perpendicular to M.

$O_{5'} \to O_3$: To see this, we first place a point Q on M and create $L = PQ$ using O_1. Now, by $O_{5'}$, fold P to L so that the fold line is perpendicular to M. The fold line must pass through P or be perpendicular to L, since P is incident to L. The fold line cannot be perpendicular to both L and M, since they meet. Thus, the fold line passes through P and is perpendicular to M.

$O_{5'} \to O_2$: Using O_1, make $M = PQ$. Now, using $O_{5'} \to O_3$, create the line L perpendicular to M at Q. Folding P to L perpendicular to M gives the fold line, which is the perpendicular bisector of PQ. □

3.1 Totally Real Constructions

In the early seventeenth century, Van Schooten discussed constructions in Euclidean geometry that depend on the transfer of lengths. In Euclidean geometry, this is equivalent to the use of angle bisectors, which gives the

Pythagorean or totally real constructions of Hilbert [Alperin 00]. Here we show the constructive power of $O_1 - O_4$ in neutral geometry.

Theorem 2. *With O_1, O_3, O_4, given center P, we can construct the symmetry or $180°$ rotation about P of any point A.*

Proof: Construct $M = PA$ and L the perpendicular to M at P; create the perpendicular from A to the angle bisectors of L, M. Construct the perpendicular from A to first bisector meeting L at A_1; construct the perpendicular from A_1 to second bisector meeting PA at A'. □

Corollary 1. *With O_1, O_3, O_4, given a line L and a point P not on L, we can construct the reflection P' of P across L. Hence the reflection of a segment PQ across L can be constructed.*

Proof: Construct the perpendicular line M from P to L, meeting it at Z; now by symmetry, using Theorem 2 about Z, we can move P to P' on M, which is the same as the reflection of P across L. For the segment PQ, we use the same construction as above for each point P and Q. □

Corollary 2. *With O_1, O_3, O_4, we can move segment of length AB to any point P on the line $L = AB$.*

Proof: Construct the perpendicular bisector M of P and A. Reflect B across M by Corollary 1 to B'. Then PB' has same length as AB. □

Corollary 3. *With O_1, O_3, O_4, we can move a segment AB to begin at any given point on any given line L through P.*

Proof: Construct M, the perpendicular bisector of PA. Reflect B across M to B'. Construct angle bisectors of L and PB'. Construct a perpendicular to the angle bisector through B_1 meeting L at B'. Then the length of PB' is the same as the length of AB. □

Corollary 4. *With O_1, O_2, O_3, the following constructions are equivalent:*

(a) a given length can be marked on any constructed ray;

(b) the angle bisector axiom O_4;

Proof: (a)$\rightarrow$ (b): Without loss of generality, we can assume the angle with vertex O is less than $180°$. We mark the given length on each ray of the angle say at A and B. Construct the midpoint of AB, then OC is an angle bisector.

(b)$\rightarrow$ (a): This is Corollary 3. □

3.2 Radical Axis and Ruler-Compass Constructions

The following construction of the radical axis works in neutral geometry; it is similar to a construction in [Handest 56]. Given two circles with centers A and B, construct the line $L = AB$. The perpendiculars to L at A and B meet the respective circles at C, D. These points are constructed using Theorem 4. The perpendicular bisector of CD meets L at P. The midpoint of AB is Z. The radical axis N is perpendicular to L passing through Q, where $ZQ = ZP$. This last can be accomplished by using a reflection, as in Theorem 2. Thus, this all can be accomplished by O_1, O_5.

Theorem 3. *All ruler-compass constructions in neutral geometry can be done using* O_1, O_5.

Proof: To see that we can do all ruler-compass constructions, we need only to show that the intersection of two circles can be constructed. By the remarks above, we can construct the radical of the two circles, and by O_5 we can construct the intersection of the radical with either of the circles to get the intersections of the two circles. □

Theorem 4. *With* O_1, O_5, *we may construct the intersection of a line and circle. With* O_1, $O_{5'}$, *we may construct the intersection of a line and an equidistant curve.*

Proof: From the previous result, we can use O_5 to create O_3. Given line L and the circle with center Q and passing through P, we use O_5 to construct the two folds reflecting P onto L with fold lines passing through Q. Now, using O_3, we drop a perpendicular from P to these fold lines which meet L at the reflected images of P.

From the previous result, we can use $O_{5'}$ to create O_3. Given line L and an equidistant curve with axis M and passing through P, we use O_5 to construct the two folds of P on L with fold lines perpendicular to M. Now, using O_3, we drop a perpendicular from P to these fold lines, which meet L at the desired points. □

3.3 Saccheri Quadrilaterals

Now we restrict our study to hyperbolic geometry. We can prove that $\mathbb{H}_0$ follows from the axioms $\mathbb{H}_1$–$\mathbb{H}_4$ using Saccheri quadrilaterals. A quadrilateral $ABCD$ with $AC \equiv BD$ and angles at A and B that are right angles is called a Saccheri quadrilateral.

Theorem 5. *In a Saccheri quadrilateral, the angles at* B *and* D *are equal (in neutral geometry) and less than* $90°$ *in* $\mathbb{H}^2$.

Proof: By the side-angle-side theorem (SAS), $\triangle ABC \equiv \triangle ABD$, so the diagonals are equal, $AD \equiv BC$. Congruence of diagonals then yields $\triangle ACD \equiv \triangle BCD$ by the side-side-side theorem (SSS); hence the angles at C and D are equal. In $\mathbb{H}^2$, triangles have less than 180 degrees so the last result now follows. □

Theorem 6. *In a Saccheri quadrilateral, the midpoint line EF is perpendicular to sides CD and AB.*

Proof: By construction, $\triangle AFC \equiv \triangle BDF$, so $CF \equiv DF$. Thus $\triangle CEF \equiv \triangle DEF$; hence, $\angle CEF = \angle DEF$ and are $90°$ degrees because they are on a line. □

This will give a construction of the common perpendicular to two (ultra) parallel lines once we establish a Saccheri quadrilateral.

Theorem 7. $\mathbb{H}_4, \mathbb{H}_3, \mathbb{H}_1 \to \mathbb{H}_0$.

Proof: Let CD be on one line and AB be the feet of the perpendiculars on the second line. In the case $AC \equiv BD$, the midpoint line EF is the common perpendicular. Use $\mathbb{H}_2$ to construct midpoints. Otherwise, one side is longer than the other. We move one in until we get equality, using the ability to transfer distances and angles, Corollary 3; now the result follows from the first case. □

3.4 Ruler-Compass Constructions

As we have shown in Section 2.4, ruler-compass constructions can be done with $\mathbb{H}_1$ and $\mathbb{H}_5$; also, it is easy to see that these can be done with ruler and compass by Theorem 3, so the constructive power of these are the same. The parallel ruler can do all ruler-compass constructions by Handest's results [Handest 56], so it is equivalent to using $\mathbb{H}_5, \mathbb{H}_1$.

However, there are other circle like objects or cycles in the hyperbolic plane, the equidistant curve, and also the horocycle. The equidistant curve is related to the use of $\mathbb{H}_{5'}$; it is a cycle where the center is outside the absolute, whereas the horocycle is a cycle with its center on the absolute.

The famous results of Nesterovich [Coxeter 47] show that the extra cycles do not add new constructive power.

Theorem 8 (Nestorovich). *Usage of other compasses (horocompass and hypercompass, which have centers on and outside the absolute, respectively, and pass through an interior point) adds no new information; that is, these constructions can be done using an ordinary (hyperbolic) compass.*

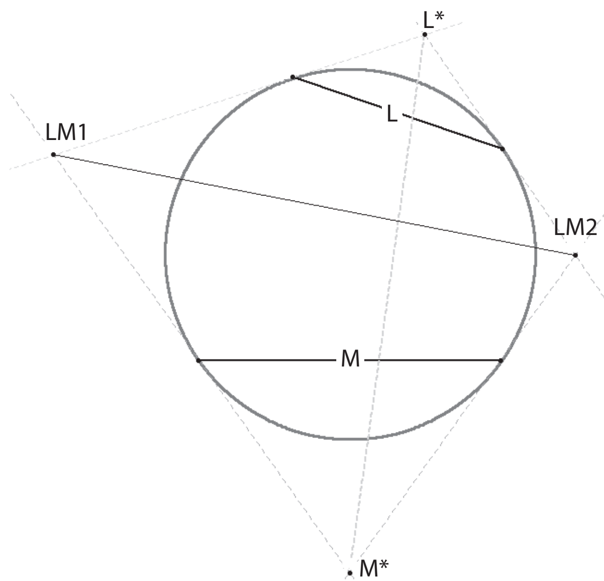

Figure 9. $\mathbb{H}_0$: common perpendiculars.

3.5 Simulation of Constructions with $\mathbb{H}_0$–$\mathbb{H}_4$

Allowing the Euclidean construction of the two tangents to the absolute at the ends of a line L, which meet at L^*, gives the facility for construction of perpendiculars to L, since any perpendicular to L passes through L^*. Note that these tangents are also perpendicular to the radial lines from the center of the absolute. We also allow the construction of the intersection of a line with the absolute.

With this extra ability, we explain how to make the non-Euclidean folds of $\mathbb{H}_0$–$\mathbb{H}_4$ on an ordinary flat piece of paper.

$\mathbb{H}_0$: Perpendicular to two lines, L, M, we construct the line L^*M^*. The other two points of intersection of these four tangent lines are LM_1, LM_2; these can be used to give the midline of L, M, thereby solving $\mathbb{H}_4$ when the lines do not meet. (See Figure 9.)

$\mathbb{H}_2$: To make the perpendicular bisector to given points A, B, we construct AL^*, BL^*, meeting the absolute in P, Q, R, S on opposite sides of L. The lines PS, QR meet L at the midpoint M of AB, and then ML^* is the perpendicular bisector. (See Figure 10.)

$\mathbb{H}_3$: Make the perpendicular to L through P by constructing the line PL^*. (See Figure 11.)

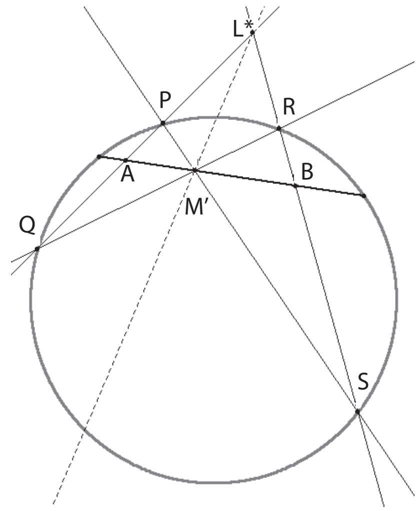

Figure 10. $\mathbb{H}_2$: perpendicular bisector.

$\mathbb{H}_4$: To make the angle bisectors of lines L, M meeting at O, first construct the ends P, Q, R, S of L, M, respectively; the lines PS, QR, PR, QS meet at O_1^*, O_2^*, respectively. The lines OO_1^*, OO_2^* are the angle bisectors. (See Figure 12.)

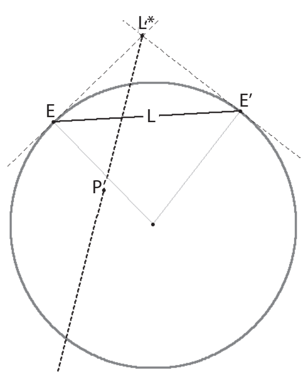

Figure 11. $\mathbb{H}_3$: perpendicular to L through P.

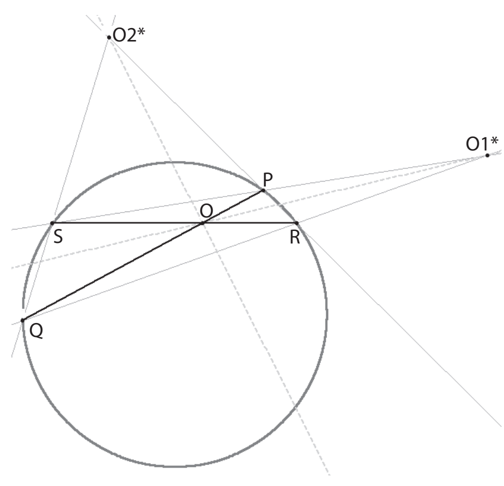

Figure 12. $\mathbb{H}_4$: angle bisectors.

4 Trigonometry and More Folding in $\mathbb{H}^2$

4.1 Hyperbolic Coordinates, Distances, Angles

For the projective model of hyperbolic geometry, the interior of the unit circle, we measure distance ρ from the origin and angle θ with respect to the x-axis.

These are the hyperbolic coordinates of $Q = (u, v)$. The distance from the origin satisfies

$$\cosh(\rho) = \frac{1}{\sqrt{1 - u^2 - v^2}};$$

hence,

$$\sinh(\rho) = \frac{\sqrt{u^2 + v^2}}{\sqrt{1 - u^2 - v^2}}, \quad \tanh(\rho) = \sqrt{u^2 + v^2};$$

also

$$\cos(\theta) = \frac{u}{\sqrt{u^2 + v^2}}, \quad \sin(\theta) = \frac{v}{\sqrt{u^2 + v^2}},$$

so in terms of the hyperbolic coordinates

$$Q = (u, v) = \tanh(\rho)(\cos(\theta), \sin(\theta)).$$

More generally, if $P = (r, s)$ and $Q = (u, v)$ are two points inside the circle, the hyperbolic distance $\rho(P, Q)$ between them satisfies

$$\cosh(\rho(P, Q)) = \frac{1 - ru - sv}{\sqrt{(1 - r^2 - s^2)(1 - u^2 - v^2)}}.$$

If line L has equation $ux + vy = 1$ and M has equation $rx + sy = 1$, then

$$\cos(\angle LM) = \frac{ur + sv - 1}{\sqrt{(u^2 + v^2 - 1)(r^2 + s^2 - 1)}}.$$

Equation of the circle. The equation of the circle centered at $O = (c, d)$ passing through $P = (r, s)$ and general point $X = (x, y)$ can be expressed simply by an algebraic relation using

$$\cosh(\rho(O, P))^2 = \cosh(\rho(O, X))^2.$$

Expanding this equation using the definition of the distance given above yields the equation of a (projective) conic,

$$x^2 + y^2 - 1 + \frac{(1 - r^2 - s^2)}{(1 - cr - ds)^2}(cx + dy - 1)^2 = 0.$$

Hence, the non-Euclidean circle with center at the origin (0,0) appears as an ordinary Euclidean circle in the projective model (where the dual of the center of the absolute is the line at infinity).

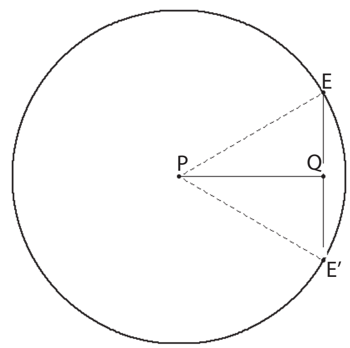

Figure 13. Asymptotic angle of 30°.

Hyperbolic trigonometry. For a right triangle with included angle θ, opposite side length a, adjacent side b, and hypotenuse c, we have

$$\sin\theta = \frac{\sinh a}{\sinh c}, \quad \cos\theta = \frac{\tanh b}{\tanh c}.$$

More complicated formulas exist for other trigonometric functions of the angles and ratios of hyperbolic trigonometric functions of the side lengths [Carslaw 16]. The analogue of the Pythagorean theorem is that

$$\cosh c = \cosh a \cosh b.$$

Angle of parallelism. The famous formula of Bolyai,

$$\cos \Pi(x) = \tanh(x),$$

relates the perpendicular distance $x = PQ$ from P to a line L (at Q) with the angle of parallelism $\Pi(x) = \angle QPE$ between PQ and the line PE where E is an end of L.

One can regard the asymptotic lines from the origin as having an asymptotic angle of 30° to the perpendicular line to the x-axis at $Q(\sqrt{3}/2, 0)$ as in Figure 13. The distance ρ is about 1.317, whereas the distance to the boundary point (1,0) from $P(0,0)$ is infinite.

Using a method of Bolyai, we may construct the asymptotic line through a given point.

Theorem 9 (Bolyai). *Given a point P not on a line L, we may construct, using $\mathbb{H}_5, \mathbb{H}_1$, the asymptotic lines through P to L.*

Proof: Construct Q on line L and construct perpendicular line M to L through P. Mark another point R on L. Construct L', the perpendicular line to M at P. Transfer distance QR to line L' as PR'. Construct M' perpendicular L at R.

Using Theorem 4, we may construct the points T, T' on the line M' which are also on the circle centered at P passing through R'. The lines PT, PT' are asymptotic (parallel) to L. □

4.2 Parallel Ruler and Its Simulation

The parallel ruler is an instrument discussed by Handest [Handest 56]. The parallel ruler constructs a line though a point P which is also parallel to a given line L, i.e. it constructs an asymptotic line (see Figure 14). In the hyperbolic plane there are two such lines. Using the result from Theorem 9, we can construct these lines using $\mathbb{H}_5$ and $\mathbb{H}_1$. Another interesting tool for ruler compass constructions is the hyperbolic ruler [Al'Dhahir 62].

It is interesting to notice that this parallel ruler is like a fold line using a modified $\mathbb{H}_1$ where we allow one of the points to be on the absolute.

We can easily simulate the parallel ruler using Euclidean constructions by intersection of the line L with the circle absolute, and then construct the lines from P to the ends E, E' of L.

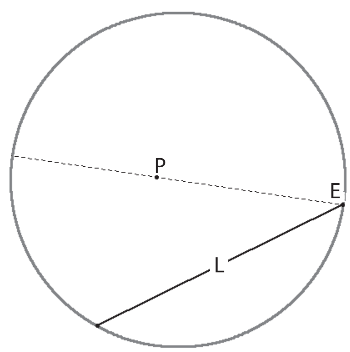

Figure 14. Parallel ruler folds a line through P parallel to L.

4.3 Theorem of Mordukhai-Boltovskoi

Theorem 10, from Mordukhai-Boltovskoi [Curtis 90], says that a segment of length x is ruler-compass constructible if adn only if (iff) $\sinh(x)$ lies in the real subfield $\sqrt{\mathbb{Q}}$, the square root closure of $\mathbb{Q}$. From this and the trigonometrical results, we see that an angle of parallelism, or any constructible angle θ, is ruler-compass constructible iff $\cos(\theta)$ is ruler-compass constructible (this is the same as the Euclidean condition).

A unit length is not constructible, since $\tanh(1)$ is transcendental. But for the points $O = (0, 0)$ and $P = (1/2, 1/2)$, we see that the distance is ruler-compass constructible since $\tanh(\rho) = 1/\sqrt{2}$ and $\cos(\theta) = 1/\sqrt{2}$. For $Q = (1/2, 0)$, since $\cosh(\rho) = 2/\sqrt{3}$, then $\tanh(\rho) = 1/2$ and $\cos(\theta) = 1$ and hence this point is also constructible.

We assume that the center O of the absolute is given.

Theorem 10. *Points in the hyperbolic plane are ruler-compass constructible iff* $\tanh(\rho), \cos(\theta) \in \sqrt{\mathbb{Q}}$*, the field of surds. Lengths are constructible iff* $\tanh(\rho)$ *is a surd. Angles are constructible iff they are Euclidean constructible; i.e.,* $\cos(\theta)$ *is obtained by using repeated square roots and field operations.*

Without using the result of Nesterovich, we can directly show how to deduce $\mathbb{H}_{5'}$ by using ruler-compass constructions (i.e., $\mathbb{H}_1, \mathbb{H}_5$).

Proposition 1. $\mathbb{H}_1, \mathbb{H}_5 \to \mathbb{H}_{5'}$.

Proof: We are given P, L, M, and we want to fold P onto L at P' with a reflection line perpendicular to M. First, construct the perpendicular from P to M meeting at Q, with PQ of length a.

Suppose that L, M meet at R, making an angle ω, and $P'R$ of length c. From Bolyai's formula for the right triangle $\triangle P'Q'R$, we have $\sin\omega = \sinh a/\sinh c$, so we can mark the distance $\sinh c$ starting from R along L to meet the equidistant curve at P'. The perpendicular bisector of PP' (or QQ') is the fold line of $\mathbb{H}_{5'}$. (Note that, to do this construction, we know that c is a constructible length since $\sinh c$ lies in the field containing $\sinh a, \sin\omega$, which is of degree a power of 2, since a and ω are constructible.)

If L, M do not meet, then first construct the intersections A and B of the common perpendicular N to lines L and M. From the unknown point P' on L construct the perpendicular meeting M at Q'. The quadrilateral $BAQ'P'$ is a Lambert quadrilateral; that is, it has three right angles. Using trigonometric formulas, we can solve for $\cosh b$, $b = BQ'$. Since we know $\sinh a$, $a = P'Q' = PQ$, and $\tanh a'$, $a' = AB$, using the $90°$ angle at B, we get $\sinh a/\sinh c = \tanh a'/\tanh c$ for $c = BP'$; hence, $\cosh c = \sinh a/\tanh a'$. Hence, by the Pythagorean theorem, we can determine $\cosh b$, so b is constructible. The perpendicular bisector of QQ' is thus the fold line needed for $\mathbb{H}_{5'}$ and can be constructed using the ruler-and-compass constructions (i.e., by Theorems 10 and 3 using $\mathbb{H}_1, \mathbb{H}_5$). □

4.4 Construction of Regular Tessellations

Suppose that we want to construct, by ruler and compass, the tessellation generated by the triangle group (p, q, r) inside the hyperbolic plane. We must have that the angles are constructible. Thus, the sines and cosines of the angles $A = \pi/p, B = \pi/q, C = \pi/r$ lie in a quadratic field. Thus,

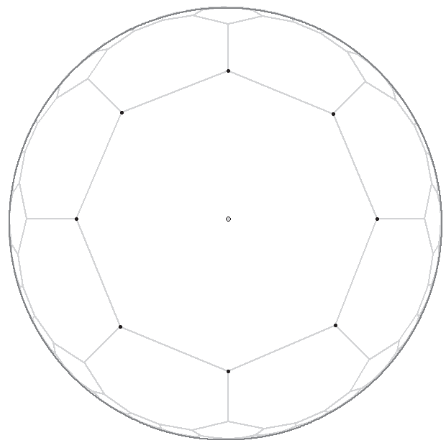

Figure 15. Tessellation of $\mathbb{H}^2$ by regular octagons.

p, q, r can be products of the Fermat primes $3 = 2 + 1, 5 = 2^2 + 1, 17 = 2^4 + 1, 257 = 2^8 + 1$, etc. and arbitrary powers of 2. These angles determine a unique triangle, so it remains only to compute the side lengths. This is easily accomplished with the hyperbolic dual triangle formula:

$$\cos C = -\cos A \cos(B) + \sin A \sin B \cosh c.$$

Hence, it follows that the cosh of the side length is already constructible since they lie in the field generated from the angles. For example to construct a (8,3,2) triangle and hence an octagon that tessellates the plane, we have $\cosh(c) = \cot A \cot B = (1 + \sqrt{2})/\sqrt{3}$, which satisfies the polynomial $9x^4 - 18x^2 + 1 = 0$. The triangle tessellation contains a tessellation by octagons that meet three at a vertex (Figure 15), so the octagons have the interior angle of $120° = 2\pi/3$. If we use other right triangles $(p, q, 2)$, we can construct tessellations by regular polygons with p sides meeting q to a vertex.

For the tessellation by right-angled pentagons, we use the $(5, 4, 2)$ triangle group. In this case, we have $\cos(\pi/5) = \sin(\pi/2)\sin(\pi/4)\cosh(c)$ and then $\cosh(2c) = (1 + \sqrt{5})/2$ corresponds to the edge length of the right-angled pentagon.

5 The Non-Euclidean Parabola

Modifying $\mathbb{H}_5$ by folding the point F onto the line L in all possible ways, we create the tangent lines to a curve. We call this the non-Euclidean parabola with focus F and directrix L (Figure 16).

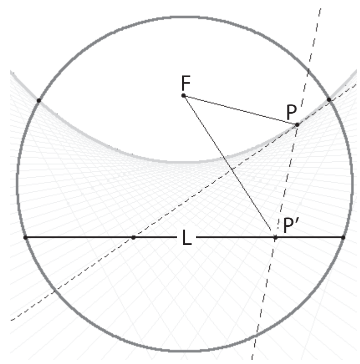

Figure 16. Non-Euclidean parabola with directrix L and focus F.

We show now that the locus $\mathcal{K}$ of points $P = (x, y)$, whose distance to the focus $F = (u, v)$ is the same as the (non-Euclidean) perpendicular distance to the directrix $L : Ax+By = 1$, is a part of a conic in the ambient plane of the absolute. We show that this curve $\mathcal{K}$ is also the envelope of perpendicular bisectors of F with points on the directrix. The tangent line to $\mathcal{K}$ at P is also the perpendicular bisector of FP', where P' is the foot of the perpendicular from P on L. The perpendicular bisector passes through P, since $PF = PP'$; any other point Q of the curve that is also on the perpendicular bisector has $QF = QQ' < QP'$, since Q' is shortest distance from Q to L. But the perpendicular bisector is the locus of X with $XF = XP'$. Therefore, there can be only one point of the perpendicular bisector on this conic; hence, it is a tangent. Thus, the locus $\mathcal{K}$ is the curve enveloped by the folds of F on L. This shows that our non-Euclidean parabola is the same as the curve discussed in [Henle 98].

Since the closest point on the directrix $Ax + By = 1$ is the intersection of the line PL^* (the perpendicular from P) and the directrix L, we use the formula for $\cosh^2$ of the distance to obtain the equation for $\mathcal{K}$:

$$\begin{aligned} K =& 1 - (u^2+v^2)(A^2+B^2) + 2x(A(v^2+u^2-1) + u(A^2+B^2-1)) \\ &+ 2y(B(u^2+v^2-1) + v(A^2+B^2-1)) \\ &+ x^2((1-B^2)(1-v^2) - u^2A^2) + y^2((1-A^2)(1-u^2) - v^2B^2) \\ &+ 2xy(AB(1-u^2-v^2) + uv(1-A^2-B^2)). \end{aligned} \tag{1}$$

Choosing special points and lines $(u, v) = (0, a)$ and $(A, B) = (0, -\frac{1}{a})$, we obtain the simplified form $4ay = (1 - a^2)x^2$. The tangents from $(b, 0)$ have a slope given by points on the dual curve, which in this case has the equation $y = (a/(a^2 - 1))x^2$.

6 $\mathbb{H}_6$

Certainly if we realize a fold line N by $\mathbb{H}_6$, $P \leftrightarrow L, Q \leftrightarrow M$, then P folds to P' on L and Q folds to Q' on M, so P', Q' are in the hyperbolic plane, since they are reflections of P, Q in N.

We know that $\mathbb{H}_6 \to \mathbb{H}_{5'}$, $\mathbb{H}_6 \to \mathbb{H}_5$ by Theorem 1 so this axiom apparently allows more than ruler-compass constructions.

Can we construct one-third of a constructible length using these origami axioms? The Euclidean origami method uses similar triangles, but they are not available in hyperbolic geometry. The problem can be restated: suppose $\sinh(x)$ is constructible, then is $\sinh(x/3)$ also? There is a familiar cubic relationship $\sinh(3x) = 4\sinh(x)^3 + 3\sinh(x)$.

Can we trisect angles? The Euclidean trisection method of Abe uses results about alternate interior angles for parallel lines, so the argument does not work in $\mathbb{H}^2$. The trisection of an angle gives a well-known cubic relationship between $\cos(\theta)$ and $\cos(\theta/3)$, coming from the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Can we solve these cubic equations $4x^3 \pm 3x - a = 0$ with the aid of origami in the hyperbolic plane? This problem is discussed in the next section.

6.1 Higher Origami Constructions

Axiom $\mathbb{H}_6$ permits folds of the common tangents to two non-Euclidean parabolas. Since these curves are conics, there are four possible tangents; these can be determined as the common points to the two adjoint or dual curves.

The equation for the dual curve of K is $K^d = 0$, where

$$\begin{aligned} K^d =& 1 + vB + uA - 2(A+u)x - 2(B+v)y \\ &+ 2(Bu + vA)yx + (1 - vB + uA)x^2 + (1 + vB - uA)y^2. \end{aligned} \tag{2}$$

To locate common tangents of two parabolas, we use the common points of the two dual curves or the resultant of the two dual curves, which is a polynomial of degree 4. The solutions to this polynomial give the information to recover the equation of the tangent lines. We can take an intersection of the adjoint or dual curves and construct the tangent lines to the given conics by using inversion. The perpendicular projection $P = \pi(L) = \tanh(\rho)(\cos\theta, \sin\theta)$ of the origin on a line L is given by inversion of the dual of L.

Now, since the intersection points of conics have their coordinates lying in the field of Vietans, $\mathbb{V}$, the real subfield of origami numbers discussed in [Alperin 00], the coordinates of P also belong to $\mathbb{V}$; hence, the sum of squares of these coordinates belongs to $\mathbb{V}$, so $\tanh\rho(O,P) \in \mathbb{V}$, and then it also follows that $\cos\theta \in \mathbb{V}$.

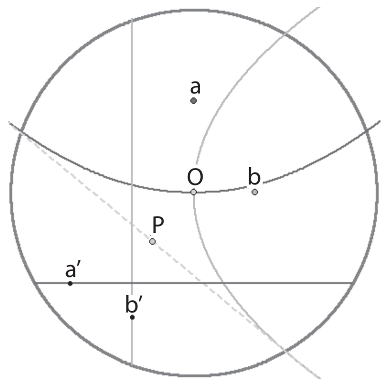

Figure 17. Using $\mathbb{H}_6$ to create $\sqrt[3]{6}$.

Theorem 11. *Any constructible length ρ or constructible angle θ using the hyperbolic origami axioms $\mathbb{H}_0 - \mathbb{H}_6$ has $\tanh\rho$ and $\cos\theta$ in the real subfield $\mathbb{V}$ of origami numbers.*

By Theorem 10, the ruler-compass constructions give the square root closure of $\mathbb{Q}$ as a subfield of the non-Euclidean origami constructible numbers. In general, the four possible tangents of two non-Euclidean parabolas have line coordinates belonging to the field $\mathbb{V}$ of Euclidean origami numbers. Thus, we can realize all Vietans if we can construct all cube roots and trisections of angles by our hyperbolic origami constructions.

We now give methods to obtain resultant equations for trisections or real cube roots.

In order to accomplish the cube roots of elements, we can use two parabolas, as in the example of Section 5. The dual equation is also the equation of a parabola. Let $(A, B) = (0, -1/a)$ and $(u, v) = (0, a)$ for the first parabola, and $(A, B) = (-1/b, 0)$ and $(u, v) = (b, 0)$ for the second parabola; then the resultant has the cubic factor $y^3 = g(a, b)$, and we can solve for values of a, b that give any positive value of $g(a, b) \in \mathbb{V}$ using $a, b \in \mathbb{V}$. For example with $a = 1/2$, $b = 1/3$, the projection of the origin on the fold line, as in Figure 17, gives a point $P = (u, v)$ with $u/(u^2 + v^2) = \sqrt[3]{6}$, where $u = \tanh\alpha$, $v = \tanh\beta$, and α, β are the lengths of the projections of P on the coordinate axes.

Because the adjoint (or dual) equations can be put in the form $X = Y^2, Y = \alpha X^2$, by suitable constructible choices of a, b, we have common solutions when $X^3 = \alpha^{-2}$; thus, we obtain the following important result.

Theorem 12. *Using the hyperbolic origami axioms, we can construct $\sqrt[3]{\tanh\rho}$ for any hyperbolic constructible length ρ.*

Using the two non-Euclidean parabolas with focus-directrix coordinates, $(u, v), (A, B)$,

$$(0, -a),\ \left(0, \frac{1}{a}\right),\ \left(-\frac{A}{A^2 - B^2}, \frac{B}{A^2 - B^2}\right),\ (A, B),$$

the resultant enables us to solve cubic equations, since these dual curves by construction have an intersection at the origin.

For the cubic $4x^3 + 3x - m$ for m in the Vietens, we use $B = 2A, a = \frac{1}{A}, A = \sqrt{r}$ and first solve for an auxiliary cube root. Namely, we solve using real cube roots and real square roots for r, so that

$$m = -\frac{3\sqrt{6}}{8}(3r + 1)\sqrt{r - 1}\sqrt{(r + 1)^3}.$$

Thus, we can trisect any segment since $\sinh(3\rho) = 4\sinh(\rho)^3 + 3\sinh(\rho)$.

For the choice of parameters $B = A/2, a = 1/A, A = -\sqrt{r}$, the resultant of the dual conics simplifies to (cf. [Alperin 05]) $4x^3 - 3x - m$ for any $m \in [.64, .84]$ in the Vietens by first solving for an auxiliary cube root, to realize m. Thus, we can also trisect angles. The somewhat intricate details of this are beyond the scope of this discussion. Basically, we need to justify only that these conic duals meet outside the absolute. For this, we develop some further properties of the dual curves; for example, when the directrices meet inside the absolute, then the conics meet only once inside the absolute; hence, there is an intersection outside the absolute and hence a solution to $\mathbb{H}_6$ that is a valid hyperbolic line.

Theorem 13. *Any point in the hyperbolic plane with coordinates in* $\mathbb{V}$ *can be constructed using the axioms* $\mathbb{H}_0 - \mathbb{H}_6$. *Hence, by hyperbolic origami, any length* ρ *or angle* θ *with* $tanh(\rho) \in \mathbb{V}, \cos(\theta) \in \mathbb{V}$ *can be constructed.*

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