Folding the Hyperbolic Crane

ROGER C. ALPERIN, BARRY HAYES, AND ROBERT J. LANG

Introduction

The purest form of origami is widely considered to be folding only, from an uncut square. This purity is, of course, a modern innovation, as historical origami included both cuts and odd-sized sheets of paper, and the 20th-century blossoming of origami in Japan and the west used multiple sheets for both composite and modular folding. That diversity of starting material continues today. Composite origami (in which multiple sheets are folded into different parts of a subject and then fitted together) fell out of favor in the 1970s and 1980s, but then came roaring back with the publication of Issei Yoshino's *Super Complex Origami* [Yoshino 96] and continues to make regular appearances at origami exhibitions in the form of plants and flowers. Modular origami (in which multiple sheets are folded into one or a few identical units that are then assembled) never really diminished at all; the kusudamas of the past evolved into the extensive collections of modulars described in books by Kasahara [Kasahara 88], Fuse [Fuse 90], Mukerji [Mukerji 07], and more.

Even if we stick to uncut single-sheet folding, however, there are still ways we can vary the starting sheet: by shape, for example. There are numerous origami forms from shapes other than square, including rectangles that are golden $(1:(\sqrt{5}+1)/2)$, silver $(1:\sqrt{2})$, bronze $(1:\sqrt{3})$, regular polygons with any number of sides, highly irregular shapes used for tessellations, circles, and more.

All of these shapes have one thing in common, however: they are all *Euclidean* paper, meaning that whatever the starting shape, it can be cut from a planar piece of paper. That means that the paper is constrained by the properties of Euclidean geometry. Those properties, in turn, determine what you can do with the paper: what sorts of distances and angles may be constructed via folding using the Huzita-Justin axioms [Huzita 89, Justin 89], and down-to-earth practical construction of 2.5-dimensional and 3-dimensional forms using algorithms such as circle-river packing [Lang 03] and Origamizer [Tachi 09] that rely on Euclidean metrics.

But what if the paper were not Euclidean?

In this work, we explore non-Euclidean origami, specifically, origami carried out with paper that possesses a uniform negative Gaussian curvature, with particular emphasis on the most iconic origami figure, the traditional *tsuru*, or origami crane. In Section 2, we define terms that connect the well-known world of non-Euclidean geometry with the less-known world of origami. Section 3 demonstrates how to create hyperbolic paper with which to fold. Section 4 analyzes and describes the desired properties of the sheet of paper from which one can fold a hyperbolic analog of the Euclidean crane, and Section 5 shows the folding of the crane itself. We close with some conclusions, some suggestions for further investigation, and instructions for folding one's very own hyperbolic, two-headed origami crane.

Euclidean and Non-Euclidean Paper

If we are relaxing the standards of so-called "purity" in origami, then we might also consider folding from non-Euclidean paper. What would this look like? Well, Euclidean paper is what is called a *developable surface*, which has the property that its Gaussian curvature is zero everywhere. A developable surface has the property that even if the paper is curled up in some way, at every smooth point, there is *some* direction in 3-space along which the paper makes a straight line, that is, the curvature in that direction is zero. A developable surface has the property that, no matter how you have bent or folded the paper, if you draw a closed curve that is always a fixed distance (measured along the surface) from any point (we call such a curve a "circle"), the length of the line is always 2π times the distance.

This isn't necessarily the case on all surfaces, though, and the most familiar counterexample is the surface of a sphere. A sphere has neither of these properties: there is no point on the surface where if you lay a pencil, the pencil touches along a continuous portion of its length. And if you start drawing circles of successively larger radii about a point on the sphere, the length of the circle does not increase linearly with the distance from the point. In fact, it can actually start to decrease; if you choose your point to be the north pole, then

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the circles around the pole are lines of latitude. As you get farther and farther from the north pole, the rate at which lines of latitude increase in length slows down and eventually reverses, with the circles becoming smaller and smaller as they approach the south pole.

The surface of a sphere has what is called *positive Gaussian curvature*. In fact, it has a property that makes it very desirable for origami: its Gaussian curvature is everywhere constant. That means if you were to cut a piece of paper from one part of a sphere and cut another piece of paper from another part of the sphere, you could place the two anywhere on the sphere with all points in contact—with each other, and with the sphere. And this would be the case no matter what the orientation of the paper might be.

Ordinary Euclidean paper has zero Gaussian curvature everywhere, so it has constant Gaussian curvature, too, and this property gives rise to the notion of *metric foldable*: the idea that one can make a fold in the paper in such a way that all of the layers on one side of the fold are in contact with all of the layers on the other side of the fold.[1]

So, a Euclidean sheet of paper has constant (zero) Gaussian curvature everywhere, and that means that it is *potentially* metric foldable; it is possible to create folds, or networks of folds, such that all of the layers of the folded result are congruent to each other and to some other zero-Gaussian-curvature surface, such as a plane, cylinder, or tipless cone. Curvature is preserved by bendings or local isometries of the surface; Gauss's *Theorem Egregium* asserts that curvature is intrinsic and can be obtained from measurements on the surface, however it may be put into space [Treiberg 08].

We can now extend the definition of metric foldability to non-Euclidean paper:

Definition 1 *A constant-curvature piece of origami is metric foldable if all layers are congruent to a surface of (the same) constant Gaussian curvature.*

This definition will clearly cover the case of a spherical piece of origami; an origami fold made from a sheet of paper cut from a sphere is metric foldable (in this sense) if the folded result can be pressed smoothly against some constant-curvature sphere—and, it's pretty clear in fact, that this must be a sphere with the same curvature as the starting sheet.

But a sphere is not the only surface with nonzero constant Gaussian curvature.

Among its many properties is the one that as you draw larger and larger circles about a point, the size of the circle increases, but sublinearly. The opposite is possible, however: there are surfaces for which if you draw circles about a point with larger and larger radii, the length of the circle increases *superlinearly*. Such surfaces give rise to a geometry known as *hyperbolic* geometry.

Hyperbolic geometry was independently invented by Bolyai and Lobachevsky. On a hyperbolic surface, the sum of the angles of a triangle is always less than 180° (on a Euclidean surface, the sum is exactly 180°; on a sphere, the sum is always greater). Hyperbolic geometry satisfies all of Euclid's postulates except the parallel postulate: given a point P not on a line L, instead of there being exactly one line through P that does not intersect L (as in the Euclidean case), there are infinitely many. This change gives rise to new properties of origami constructions when undertaken in hyperbolic geometry (see [Alperin 11]). But it can also give rise to new properties in folded origami design. Hyperbolic geometry is the geometry of a complete, simply connected surface of constant negative Gaussian curvature.

A surface has *negative* Gaussian curvature at a point if the surface curves down in one direction and curves upward in the perpendicular direction. Or, in other words, it is shaped like a potato chip. An example of such a surface is the *hyperbolic paraboloid* shown in Figure 1, which has the simple equation

$$z = x^2 - y^2. \quad (1)$$

The hyperbolic paraboloid is familiar to many origami enthusiasts, as that is the name of a fairly well-known origami model composed of pleated concentric Waterbomb Bases, recently proven by Demaine, *et al.* to "not exist" (at least, not without the addition of extra creases) [Demaine, *et al.* 11].

AUTHORS

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[1] Metric foldability is similar to the notion of flat foldability common in the origami literature (see, e.g., Hull [Hull 02]), but flat foldability typically encompasses both metric conditions and conditions on layer ordering that enforce non-self-intersection. Metric foldability does not consider questions of self-intersection, and so is a somewhat weaker condition.

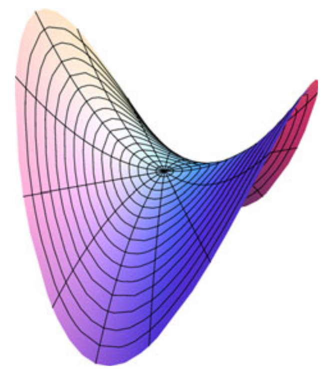

Figure 1. Plot of a hyperbolic paraboloid, a model of nonconstant negative Gaussian curvature.

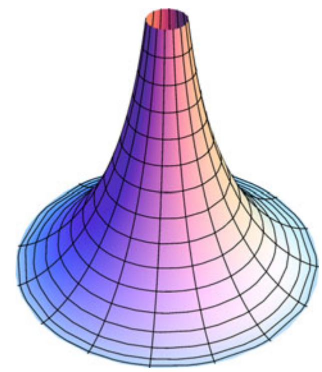

Figure 2. Plot of half of a pseudosphere (truncated at the top).

One could imagine making a sheet of paper of this shape (perhaps by dampening a potato chip until it gets soft?) and folding something from it, but the hyperbolic paraboloid has a problem: it has negative Gaussian curvature, but it does not have *constant* Gaussian curvature. In fact, its curvature is highest in the very middle and then drops off toward the edges. As one proceeds farther and farther from the center, the value of the curvature becomes smaller and smaller, and so the paper becomes more and more Euclidean away from the middle. The problem with nonconstant Gaussian curvature is that the paper cannot be folded congruently to itself except along a few special lines of symmetry. We need a surface with constant negative curvature, so that fold lines can, in principle, run any direction through any point. So the hyperbolic paraboloid does not work as a starting surface for non-Euclidean origami.

To obtain a surface that has constant negative Gaussian curvature, we turn to another model: the *pseudosphere*, so called because (like a sphere) it has constant Gaussian curvature, but (unlike a sphere) its curvature is negative. The pseudosphere has the equation

$$z = \pm \operatorname{sech}^{-1}\sqrt{x^2+y^2} - \sqrt{1-(x^2+y^2)}, \qquad (2)$$

or a parameterization $f_{\pm} : [0,\infty) \times [-\pi,\pi) \to \mathbb{R}^3$ given by

$$f_{\pm}(u,v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \pm(u - \tanh u)), \qquad (3)$$

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and its top half (f_+, a hemipseudosphere) is shown in Figure 2; the bottom half (f_-) is its mirror image in a horizontal plane. Going forward, we will only be concerned with the upper hemipseudosphere.

A pseudosphere has constant negative curvature. Because it is constant, if you cut a patch of a pseudosphere, you can slide it around on the surface of the pseudosphere keeping all points in contact with the pseudosphere at all times. You can turn it around, slide it up and down, anything you like—but you can't cross the sharp rim, where there is a singularity in the curvature. The sliding patch may, as it slides about, overlap itself: if you cut a large patch from near the base and then slide it upward, the higher it goes, the more tightly it will wrap around the central spire until, eventually, it will begin to overlap itself. And of course, it must bend as it slides around (unlike spherical patches, which can slide on a sphere without bending). But the important thing is that pieces of pseudosphere can slide on a pseudosphere without stretching. And that means that we can apply our notion of metric foldability to pseudospherical, hyperbolic paper.

Pseudospherical Paper

One can, of course, analyze origami with hyperbolic paper in purely mathematical terms, but what makes this problem interesting is the practical possibilities: we can, actually *make* hyperbolic paper and carry out folding experiments with it.

To do so, we need two things: (a) a hyperbolic sheet of paper, (b) a hyperbolic desk to fold on.

Well, we don't actually *need* the second part. We can fold Euclidean paper in midair (in fact, origami master Yoshizawa has famously advocated midair folding for all folders, although video evidence reveals a suspiciously desk-like folding surface under his own paper at times[2]).

When folding in midair, one creates a fold by pinching the paper at a point in the direction of the fold and then continuing to flatten the pinch along a line without twisting or changing the direction. This is the same as folding along a *geodesic* of the surface, a curve on the surface whose nontangential component of acceleration is zero. But physically performing this manipulation in midair without unintentionally incorporating small lateral accelerations is difficult,

[2]See, for example, the 2009 documentary film *Between the Folds*.

and if the fold is not along a geodesic, the layers on opposite sides of the fold will not lie flat against one another. The result will not be metrically flat.

With Euclidean paper, it is much easier to create a metrically flat fold if, after we have begun a fold, we press it flat against a Euclidean flat table. With non-Euclidean paper, we will also wish to press the paper "flat" after folding, but for non-Euclidean paper, "flat" means "congruent to a surface with the same Gaussian curvature as the paper." Thus, we will fold our pseudospherical paper while seated at a pseudospherical desk, against which we will fold.

Even before making a fold, however, we must acquire pseudospherical paper. A quick check of the local art stores revealed no pseudospherical paper to be found. We must make it ourselves.

The usual way of making paper by hand uses a mesh screen, which is dipped into a vat of paper slurry, then is lifted out; the water drains through the mesh, leaving behind a thin layer of paper fibers on the mesh. This layer is then inverted onto a felt; a stack of paper/felt layers is pressed to mat the fibers and to remove water; and then the resulting layers are stacked between layers of blotter paper and cardboard for further pressing and drying. Commercial paper making follows a similar process, but uses mesh belts and rollers to realize a continuous manufacturing process.

That presents a whole slew of problems for making hyperbolic paper, beginning with the very first step. When one lifts the mold out of the paper slurry, it is critical that the mesh surface remain exactly horizontal; even the slightest tilt creates variations in paper thickness or even creates holes, as the paper/water slurry runs off the side, rather than the water draining evenly through the mesh.

But it is impossible to make a hyperbolic mesh that is horizontal at every point; if it's horizontal at every point, then you have a Euclidean plane. And since just a few degrees of tilt is enough to ruin a sheet of paper, there would be no way to pull a hyperbolic sheet with a lot of curvature—even if one had a hyperbolic mesh.

And, of course, the hyperbolic form would need to be replicated in all of the other elements of the paper-making process. Sounds expensive.

There's another way, though: we could, perhaps, start with a sheet of Euclidean paper and modify it to make it hyperbolic.

How would we modify it? Well, we'd have to give it negative curvature everywhere. We'd have to give it the property that, if you draw concentric circles, the ones farther away become longer at a faster rate than linearly. So we could start with a sheet of paper, fix one point, and then, as we move farther away, stretch it out so that it becomes larger and larger, doing the stretching at exactly the right rate to keep the negative curvature constant.

This is, in fact, precisely the approach taken by textile artists who knit or crochet hyperbolic surfaces (see, for example, [Taimina 09], [belcastro and Yackel 07]). By crocheting in concentric circles and strategically adding stitches as the radius increases, one can obtain the required increase in length, which gives the desired curvature of the resulting surface. This only works for finite regions of the hyperbolic plane; as Hilbert showed (1901) [Treiberg 03] there is no geodesically complete smooth immersion of the hyperbolic plane in $\mathbb{R}^3$. However, one can view the crocheted surfaces as confirmations of the results of Kuipers and Nash (1955–1956) [Treiberg 03] that by sufficiently crinkling the hyperbolic plane one can obtain an embedding into $\mathbb{R}^3$.

One could take a similar approach to create an approximation of a hyperbolic surface with paper. One could, for example, take a sheet of Euclidean paper, cut slits into it, and insert strategically chosen wedges to give the desired increase in circular length as one moves out from the center. Or, for a more scalable approximation, one could create a hyperbolic tiling, joining equilateral triangles with seven triangles at each vertex, or a tiling of hexagons and heptagons with two of the former and one of the latter at every vertex. There are many possibilities of this sort, but they all suffer from a flaw: they are only approximations of hyperbolic paper. Locally, each point is either Euclidean (in the interior of a tile) or singular (at a vertex); it is only the large-scale average behavior that approaches a true hyperbolic surface. What we'd really like to do is distribute the extra length evenly, rather that insert it in discrete "chunks" within the surface.

Even distribution of extra length would imply stretching the paper out continuously at its edges as you move farther and farther away from a central point. The problem there is that paper doesn't stretch; it rips.

But, one thing paper does do fairly well is "bunch up." Instead of fixing the center and stretching the edges, we could fix the edges and "bunch up" the center; that would have the same effect. The amount of bunching is critical, though; it needs to be just the right amount at every point of the paper. Too much, and the curvature is too large at that point; too little, and the curvature is too small. And if the curvature is nonuniform, then one loses the ability to do metrically flat folding. We need to introduce just the right amount of curvature at every point so that the result is congruent to the hyperbolic desk that we are folding against.

So, let's use the "desk" as a mold; we will take a Euclidean sheet of paper and mold it against a hyperbolic form; the result will be a pseudospherical sheet of paper from which we can do true hyperbolic origami.

But where do we get a hyperbolic desk? In the old days, this would have been a simple thing to obtain. The pseudosphere is a solid of rotation; its cross section is the *tractrix*, which has the equation

$$z = \pm \operatorname{sech}^{-1} r - \sqrt{1 - r^2}, \tag{4}$$

One could simply create a template of this curve, then give it to a handy woodworker who could turn it on a lathe. But nowadays, this is an even simpler thing to obtain. Thanks to the existence of commercial 3D printing companies such as Shapeways (http://www.shapeways.com), one can generate a 3D model in *Mathematica*™, upload it to one's website, and a few weeks later (and a few dollars poorer), one receives a plastic 3D model of the desired figure.

Now, using the hyperbolic form, we create the sheet of hyperbolic paper. Take a sheet of thin, strong kozo paper and saturate it with a solution of Carboxymethylcellulose (a sizing agent) and drape it loosely over the pseudosphere. Although it will stick near its edges, this leaves enough excess paper in

Figure 3. Left: the plastic pseudosphere. Right: a sheet of pseudospherical paper after forming and drying.

Figure 4. The pseudospherical sheet with a single fold.

the middle so that it can be smoothed down against the pseudosphere, where the excess paper will gather into irregular pleats across the surface. By smoothing it down completely, it sticks to the surface all the way up, down, and around the form (with the excess paper gathered into a rough flange sticking out from the back). Now let the whole thing dry.

After the paper dries, the CMC bonds all of the layers of the pleats firmly together, so that it all behaves like a single sheet. After cutting off all excess that isn't stuck to the pseudosphere (including the flange on the back side), we have a single sheet that is congruent to the pseudosphere, extending nearly all the way around it (there will be a gap where the excess-paper flange is cut away). Then peel the paper off of the pseudosphere. The result is a sheet that has some small variation in thickness over its surface, but has a uniform negative curvature that exactly matches that of the pseudosphere. The mold and resulting sheet are shown together in Figure 3.

We now have hyperbolic paper. With a concrete object in hand (well, a cellulose object), it is time to do some folding.

Hyperbolic Folding

What does a fold look like on hyperbolic paper? Or, to be precise, what does a *metric* or flat fold look like? Can any (possibly curved) line on a hyperbolic sheet be folded?

We can address this by analogy with Euclidean paper. On a Euclidean sheet, for a fold to be folded flat, it must be straight. It is possible to create a fold line that is curved within the surface, but if this done, then as the fold is formed, the paper on either side of the fold takes on a curved shape in 3D, and it is not possible to close the dihedral fold angle completely. In fact, the two-dimensional curvature of the fold line and its 3D curvature are linked by the dihedral angle of the fold itself [Fuchs and Tabachnikov 99]; as the dihedral angle approaches 180°, the 3D curvature of the fold line becomes infinite, unless the fold line itself is straight. So in Euclidean paper, all fold lines in a flat fold must be straight between vertices.

Exactly the same condition must apply with hyperbolic paper. If we "blow up" a patch of hyperbolic paper to a high magnification, it begins to look locally flat, and so the same local laws as metric flat folding must apply to infinitesimally small regions of hyperbolic paper. So on hyperbolic, as well as Euclidean paper (and spherical paper, too, for that matter), all metric-foldable fold lines must be straight. Specifically, they must be geodesics.

If we fold along a geodesic, then we can achieve a hyperbolic metric fold: the folded layer can lie back on itself congruently, so that it touches at all points. An example of this is shown in Figure 4.

Figure 4 shows the paper after making a single (vertical) fold, and it is obvious from symmetry considerations that the two layers of the paper must be congruent with each other. What is less obvious, but follows directly from the constant Gaussian curvature, is that this fold and the double-layered paper could be wrapped around the pseudosphere congruently, in any rotational orientation, without stretching. And, in fact, any metric-fold could be similarly wrapped.

So, we have hyperbolic paper, and we can make folds in it. We can make hyperbolic origami. But what hyperbolic shape should we make?

Ideally, we should fold something that exploits the hyperbolicity in some special way. There's not much point in folding something that looks pretty much the same as what you get with Euclidean paper. We'd like to show off some property that comes as part of the hyperbolic geometry.

And one property that comes along with hyperbolic geometry is that there is no such thing as similarity: the geometric properties of polygons depend on the specific size of the polygon. The sum of the angles in a triangle is less than 180° in hyperbolic space, but it's not a fixed number; in fact, it can be made as small as desired, simply by making the triangle larger and larger.

Conversely, if instead of fixing the number of sides, we fix the corner angles, we can increase the number of sides in a polygon simply by making it larger and larger. If we have a quadrilateral whose corner angles are some value α, then there is a somewhat larger pentagon whose corner angles are α, and a still larger hexagon whose corner angles are α, and so forth and so on, with no limit.

This is a very nice property! We can make use of it when we get down to the origami problem of constructing a *base*—a geometric shape that has the same number of flaps as the subject has appendages.

With Euclidean paper, if we want to make a base that has N equal points coming from the edges of the paper, we must use an equilateral N-gon; each corner of the N-gon becomes a

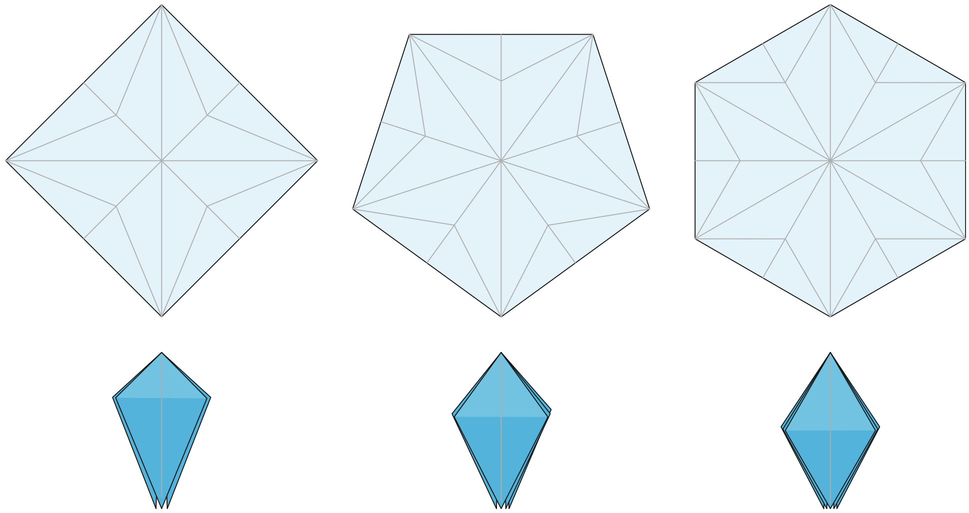

Figure 5. Top: Crease patterns for $N = 4, 5, 6$-gons. Bottom: The folded origami bases. Note that the flaps (shaded) become shorter and wider as N increases.

point. The aspect ratio of each point is related to the corner angle of its polygon; the larger the corner angle, the shorter and wider each point becomes. Thus, as N increases, the corner angles increase, approaching 180°, and, for a given number of layers in each point, the individual points become shorter and stubbier. Furthermore, a larger fraction of the paper goes into the interior of the N-gon relative to the amount used for the edge points, leading to wasted paper and thickness in the base. Figure 5 illustrates this shortening for $N = 4, 5, 6$-gons. The $N = 4$ shape is the classic Japanese form called the "Bird Base," which is the basis of the traditional *tsuru*, as well as many other origami figures both traditional and modern.

In an edge-point base, if we want to fix the aspect ratio of the points of the base, we need to preserve the corner angle of the polygon from which the base is folded. Going from 4 to 5 points by going from a square to a pentagon will shorten and widen all 5 points. John Montroll recognized this tradeoff and devised an innovative solution to the problem decades ago: his "Five-Sided Square" [Montroll 85]. Imagine dividing a square into quarters along its diagonals, so that it appears to be composed of four isoceles right triangles. Now, imagine splicing in a fifth isoceles right triangle. This results in a shape that is an equilateral pentagon that has five right-angled corners. Such a shape does not exist in the Euclidean plane, and indeed, Montroll's "Five-Sided Square" cannot be flattened into a single layer. But it does solve the problem of getting five points whose aspect ratio is the same as those obtained from a Bird Base—the basis of the traditional Japanese *tsuru*, and a host of other origami designs.

Montroll's "Five-Sided Square" is not a hyperbolic surface; it is a Euclidean surface with a single singular point at its center. (Montroll's brilliant innovation was to find a way to fold this shape from a single uncut Euclidean square.) But now that we have hyperbolic paper in our arsenal, we can do exactly the same thing without folding. Hyperbolic paper allows one to cut a *real* "Five-Sided Square"—or, rather, not precisely a square, but a real five-sided equilateral polygon whose corner angles, like those of a Euclidean square, are all right angles.

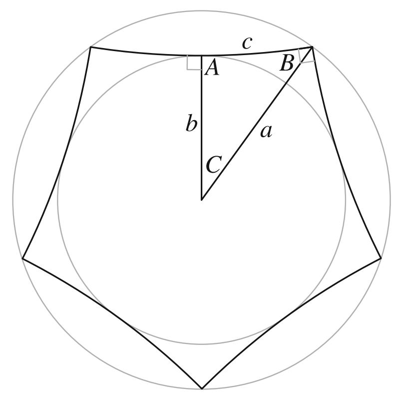

Figure 6. Schematic of a hyperbolic pentagon with inscribed and circumscribed circles.

In fact, we can cut an equilateral/equiangular right-angled polygon with any number of sides from a sufficiently large sheet of hyperbolic paper. But not necessarily from a sheet formed on a hemipseudosphere, which has finite size.

We should work out how much paper a right-angled polygon requires. A right-angled N-gon can be broken up into $2N$ triangles with corner angles $\{\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{N}\}$ radians, as illustrated in Figure 6 for a hyperbolic pentagon.

The area of a triangle $\triangle$ with angles A, B, C in hyperbolic geometry is given by the excess formula

$$\text{Area}(\triangle) = \pi - A - B - C. \qquad (5)$$

So the right-angled polygon with N sides, consisting of $2N$ such triangles, has area $2N\pi(1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{N}) = \pi(\frac{N}{2} - 2)$, which increases linearly with N.

The surface area of the unit pseudosphere is finite, however; in fact, it is 4π (the same as that of a real sphere with the opposite Gaussian curvature), so there is some upper bound on the number of sides of a right-angled polygon cut from a hemipseudosphere. We cannot cut an arbitrarily large polygon from a sheet of finite area (even setting aside the fact that our mold with the top spine truncated is not even half of a

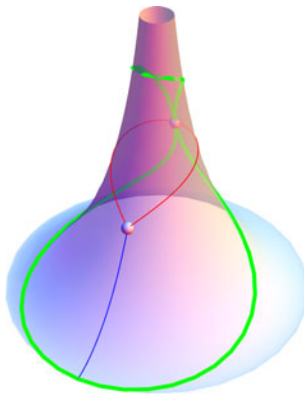

Figure 7. A hemipseudosphere with the largest nonoverlapping circle (green). The red and blue lines are radii of the circle.

pseudosphere). A right-angled pentagon with area $\frac{\pi}{2}$ might well fit in the upper hemipseudosphere with area 2π, but the right-angled octagon, with area 2π but a different shape, certainly wouldn't.

We can tighten up these bounds a bit by considering the largest circle on a hemipseudosphere, which touches the bottom of the hemipseudosphere on one side and touches itself on the opposite side as it wraps around the central post, as illustrated in Figure 7.

The radius of this circle can be analyzed by solving for the point that equalizes the lengths of the three radii shown in Figure 7; all three lines must be geodesics. If we use the (u, v) parameterization of the surface that was introduced in Eq. 3 and we denote the center of the circle by (u_c, v_c), then its height above the bottom rim, measured along the vertical geodesic (which is the tractrix already mentioned), is

$$d_{lower} = -\frac{1}{2}\log[1 - \tanh^2 u_c]. \tag{6}$$

The two upper radii touch on the opposite side of the hemipseudosphere at the point where the circle touchs itself. Being radii, each must be perpendicular to the circle at the point of tangency; the two circles are, by definition, collinear at the point of tangency; thus, the two radii are collinear with each other and consequently must be part of the same geodesic.

On a hemipseudosphere, all nonvertical geodesics can be parameterized in terms of (u, v) as

$$(u, v) = (\sinh^{-1}[k \sin\phi], c + k\cos\phi), \tag{7}$$

where $k \in (0, \infty)$ and $c \in [0, 2\pi)$ are parameters that characterize the particular geodesic and $\phi \in [0, \pi]$ gives the position along the geodesic. k is a measure of the "steepness" of the geodesic; c measures its rotational position about the center spine. For k sufficiently large, a given geodesic can cross itself one or more times; the geodesic that provides the two upper radii in Figure 7 must be one of those, where the point of tangency of the circles is the high point of the geodesic (at $\phi = \pi/2$) and the circle center (u_c, v_c) is the first crossing below the high point.

Any two points where a geodesic crosses itself must have the same value of u, which means, from Eq. 7, that they must have the same value of $\sin\phi$; this, in turn, means that the ϕ-values at the two points on the geodesic must be a pair $\{\phi, \pi - \phi\}$. Their v values must differ by some multiple of 2π, which means that the possible values of ϕ at a crossing point can take on only discrete values:

$$\phi_c = \cos^{-1}\frac{n\pi}{k}, \tag{8}$$

for some integer n. The $n = 0$ solution gives $\phi = \pi/2$, which is the high point of the geodesic; the crossing we are after is the next one down, the $n = 1$ crossing, which will have $\phi_c = \cos^{-1}\pi/k$. Knowing that this geodesic passes through the point (u_c, v_c), we can solve for the parameters of the geodesic, finding

$$k = \sqrt{\pi^2 + \sinh^2 u_c}, c = v_c - \pi, \ \phi_c = \cot^{-1}[\pi \csc u_c]. \tag{9}$$

Now, we can measure the distance from the crossing point $(\phi = \phi_c)$ to the high point $(\phi = \pi/2)$ along the geodesic; that should be the length of one of the upper radii. Carrying out the integration, we find that

$$d_{upper} = \coth^{-1}[\pi \text{ sech } u_c]. \tag{10}$$

Now we can set the two distances equal, $d_{lower} = d_{upper}$, and solve for u_c, yielding

$$u_c = \log[\sqrt{2\pi} + \sqrt{1 + 2\pi}], \tag{11}$$

and thus,

$$d_{lower} = d_{upper} = r_{max} = \frac{1}{2}\log(1 + 2\pi) \approx 0.992784 \tag{12}$$

is the radius of the largest possible circle that can be cut from a unit hemipseudosphere without overlap.

Now, with respect to Figure 6, we can solve for the radius of the incircle and circumcircle of a 90° N-gon by making use of the Sine Law for hyperbolic geometry,

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}, \tag{13}$$

the Cosine Law for hyperbolic geometry,

$$\cos C = -\cos A \cos B + \sin A \sin B \cosh c, \tag{14}$$

and the corner angles given previously. Solving all of these, we find that the circumcircle radius, the incircle radius, and semiside length are, respectively,

$$a = \cosh^{-1}\left[\cot\frac{\pi}{N}\right],$$
$$b = \text{sech}^{-1}\left[\sqrt{2}\sin\frac{\pi}{N}\right],$$
$$c = \cosh^{-1}\left[\sqrt{2}\cos\frac{\pi}{N}\right].$$

Numerical values of these are summarized in Table 1.

Table 1. Circumcircle radius (*a*), incircle radius (*b*), and semiside length (*c*) for 90° *N*-gons

N	*a*	*b*	*c*
5	0.84248	0.62687	0.53064
6	1.14622	0.88137	0.65848
7	1.36005	1.0704	0.72454

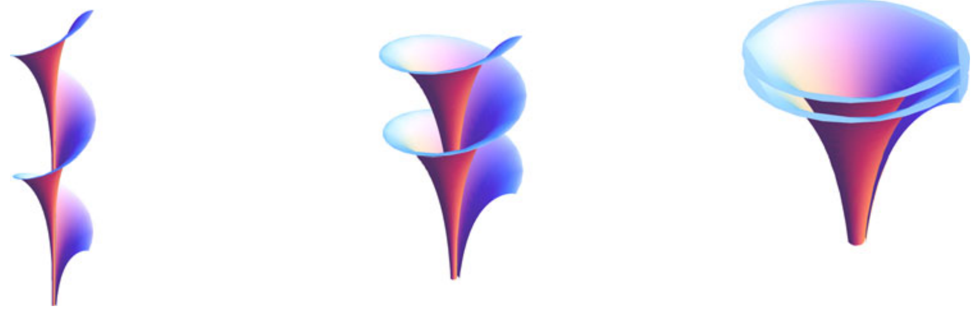

Figure 8. Three segments of Dini's surface, for $t = 0.5$, $t = 0.2$, and $t = 0.03$, respectively.

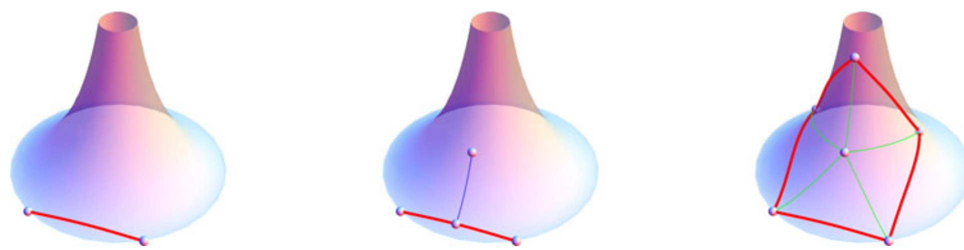

Figure 9. Stages of construction of a hyperbolic pentagon. Left: construct the base. Middle: move vertically to find the center point. Right: extend outward to locate the 5 corners.

For a given polygon, if $a \leq r_{max}$, then the polygon fits inside the largest nonoverlapping circle and so can definitely be cut from a hemipseudosphere. Conversely, if $b > r_{max}$, then the polygon fully encloses that largest circle and definitely cannot be cut from a hemipseudosphere. From this argument and the numbers in Table 1, we can see that a pentagon is definitely possible; a heptagon is definitely impossible; and a hexagon may or may not be possible—we still don't know enough to determine for sure.[3]

Now, a pseudosphere is not the only 3D surface with constant negative curvature. There is, in fact, an entire family of such surfaces, known as Dini's surface [Weisstein 04]. The family of Dini's surfaces with Gaussian curvature equal to -1 can be described by a parameterization $f_t : (-\infty, \infty) \times (0, \pi/2] \to \mathbb{R}^3$, characterized by a parameter t, with

$$f_t(u, v) = (\cos t \cos u \sin v, \cos t \sin u \sin v, \cos t\left(\cos v + \log \tan \frac{v}{2}\right) + u \sin t\Big). \quad (15)$$

Three examples of Dini's surface are shown in Figure 8 for three different values of the parameter t. The surface is infinite in both the $+z$ and $-z$ directions and can, in fact, accommodate an arbitrarily large circle (and thus, a regular 90° polygon with any number of sides). However, most of the circle ends up wound tightly around the downward-pointing spine, which presents some distinct practical challenges to paper-making with Dini's surface as a mold.

However, since we do know that a 90° pentagon is possible on the hemipseudosphere, we can now proceed with its construction as shown in Figure 9.

To construct the 90° pentagon, we first construct a geodesic whose total length, rim-to-rim, is given by the side length calculated from the expressions previously described, that is, a side length of $2\operatorname{sech}^{-1}\sqrt{3-\sqrt{5}}$. The endpoints of this geodesic must be two corners of the pentagon. These are illustrated on the left in Figure 9.

Then, from the midpoint, we can travel up the hemipseudosphere by a distance equal to the incircle radius, which is $\operatorname{csch}^{-1}\sqrt[4]{5}$, as illustrated in the center subfigure. This will be the center of the polygon.

Finally, we extend outward five rays, equally spaced in angle, by a distance equal to the circumcircle radius, which is $\sinh^{-1}(\sqrt{2}/\sqrt[4]{5})$. These give the five corners of the pentagon (recovering the two original corners we started with). Connecting all five gives the complete 90° pentagon.

Using a similar approach, we can attempt to construct a 90° hexagon. However, we will find that two corners overlap on the back side of the hemispseudosphere. This is not the only orientation to consider; one could rotate the hexagon about one of the corners of the rim, but for every possible orientation, there remains some overlap, as illustrated for two rotation angles in Figure 10. Thus, the 90° pentagon is the unique 90° polygon that can be cut from a hemipseudosphere.

Armed with this knowledge, we can now turn to the practical problem: how do we actually cut a piece of paper to a right-angled pentagon with equal sides and angles?

The easiest way to do this is to fold the paper into tenths about a point, so that one can cut through all ten layers at once. With the paper folded in this fashion, one must choose the cut location so that the base angles of the resulting triangle are, respectively, 45° and 90°, per Figure 6. The resulting pentagon (and its negative, the hole left behind), are shown in Figure 11.

[3] A more detailed analysis, beyond the scope of this article, reveals that there is, in fact, no nonoverlapping 90° hexagon on the hemispseudosphere.

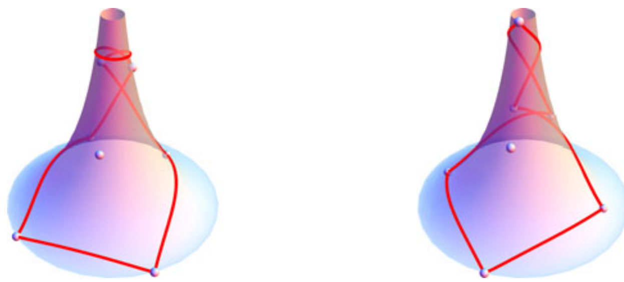

Figure 10. Left: a 90°-hexagon with two corners on the rim. Right: the same hexagon with one corner on the rim and symmetry position.

Of course, both the pentagon and the leftover paper remain congruent to the hemipseudosphere on which they are formed.

Folding the Hyperbolic Crane

And now, with a true five-sided hyperbolic square, we can fold a shape that would normally come from a square, but the result will now have five, not four, flaps. So we can fold the traditional Japanese Bird Base and from that, fold a traditional crane, but with a twist. Or rather, not a twist, but with an extra corner. That corner translates into an extra flap, which needs to go somewhere. We have chosen to turn it into a second head. Thus, a hyperbolic crane, with the same head and wing angle as the traditional crane, has two heads.

A completed Japanese *tsuru* is a 3D fold, even from Euclidean paper, but each stage of its folding is a flat fold in between 3D manipulations. The same goes for the hyperbolic crane: individual folding steps take it through 3D manipulations, but each stage of the folding is metric flat, in a hyperbolic sense; meaning that each stage can be pressed congruently against the hemipseudospherical desk. Two such stages, corresponding to the analogs of the shapes known as the Preliminary Fold and The Bird Base, are shown in Figure 12.

The Euclidean crane is traditionally displayed with its layers spread in 3D. Our hyperbolic crane can be similarly shaped for display embedded in $\mathbb{R}^3$, and we show the result, which we believe is the first example of hyperbolic decorative origami, in Figure 13.

It looks pretty much like the traditional crane (aside from the extra head), but close examination will reveal a slight double-curvature of the wings, arising from the negative Gaussian curvature. In fact, every such surface is similarly double-curved, but at this scale, the curvature is hard to perceive.

Folding instructions for the traditional Euclidean crane are to be found in many basic origami books and online sources (see, e.g., [Kirschenbaum 11]), but folding instructions for hyperbolic cranes are less commonly found. We present folding instructions as follows for readers who wish to try to fold their own two-headed hyperbolic crane.

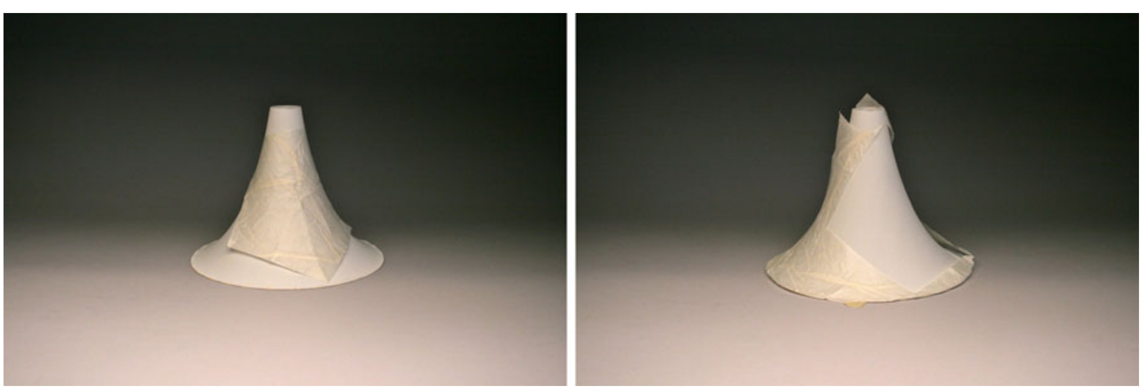

Figure 11. Left: The pentagonal paper. Right: The hole left behind.

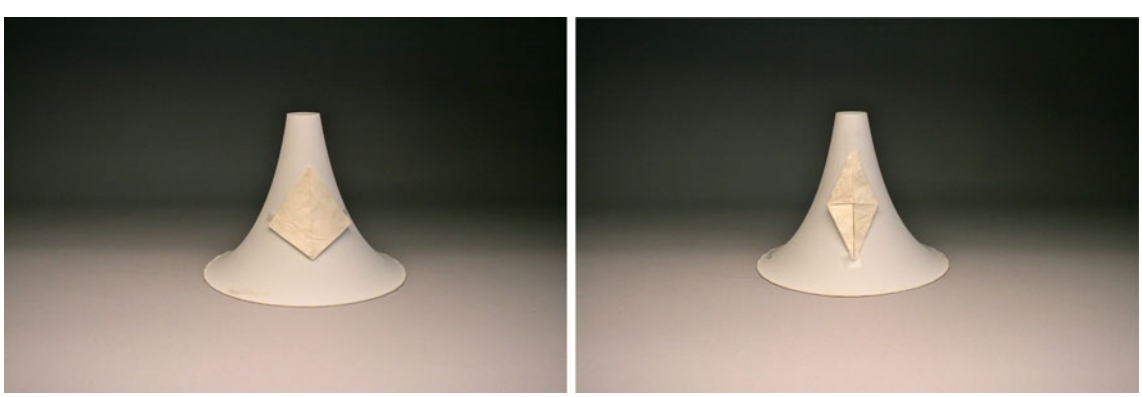

Figure 12. Left: Hyperbolic analog of the Preliminary Fold. Right: Hyperbolic analog of the Bird Base.

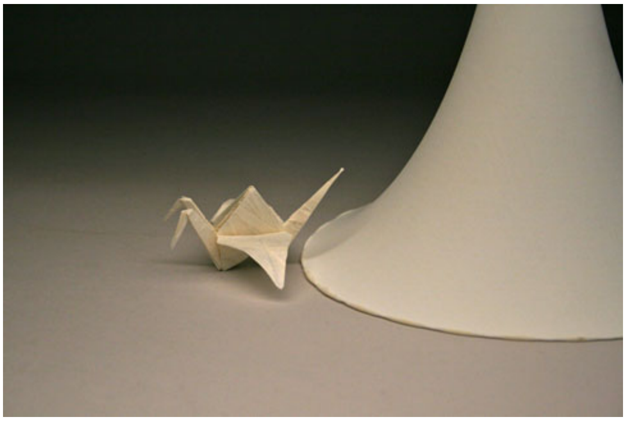

Figure 13. The hyperbolic crane.

Conclusions

Folding a hyperbolic crane (or hyperbolic origami in general) is a satisfying and tangible way of exploring some of the properties of hyperbolic surfaces and geometry. It is clearly possible to create a five-limbed analog of the traditional Japanese *tsuru*; in fact, with sufficiently large paper, one could create any number of limbs, though one might need to splice together multiple sheets formed on a pseudospherical mold. The act of folding, though, raised several questions for further study:

- Do the intermediate stages between flat-folded steps exist? That is, is there a deformation in $\mathbb{R}^3$ from one flat-folded step to the next that creates no new creases (smooth bending only between folds) and that does not stretch the surface?
- Does the 3D finished crane exist? That is, is there an embedding in $\mathbb{R}^3$ of a deformation of the flat-folded crane that resembles Figure 13 that involves no stretching or new creases beyond those that exist in the 3D embedding of the Euclidean crane?

Based on the fact that it was possible to fold the hyperbolic crane, it seems plausible that the answer to both questions is "yes," but this is by no means assured, as the imperfections and "give" of the paper could have easily masked subtle mathematical impossibilities during the formation of the paper shape.

Beyond those two specifics, numerous additional questions arise. Traditional origami includes not just single folds, but combinations of folds: "reverse folds," "rabbit-ear folds," "sink folds," and more [Lang 03]. Some of these are possible to form without creating new creases; some, such as "closed-sink folds," are not. Does hyperbolic paper allow more, fewer, or exactly the same family of combination folds without creating additional creases? These and other questions offer fruitful opportunities for future research and exploration.

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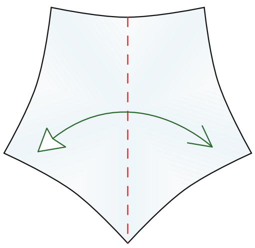
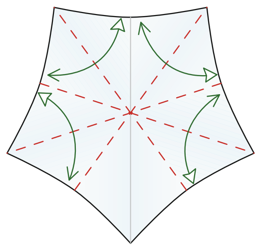
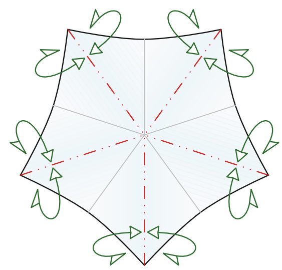
Hyperbolic Crane



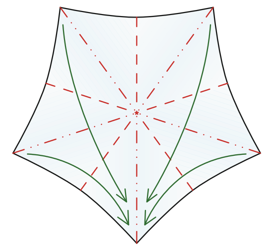
1. Begin with a hyperbolic pentagon. Fold in half through one corner and unfold.



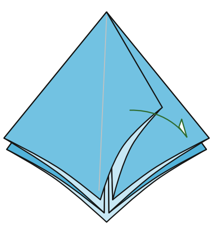
2. Repeat with each of the other 4 corners.



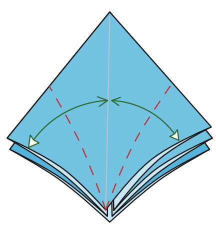
3. Change each of the creases that run from corner to center to mountain folds, i.e., crease each fold in the other direction.



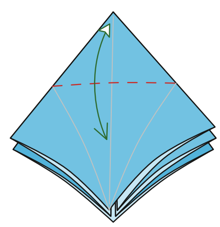
4. Using the existing creases, gather all 5 corners together at the bottom.



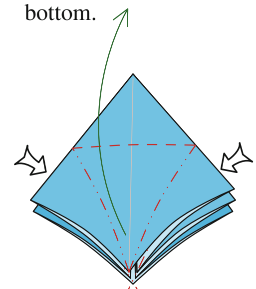
5. There is one extra flap in the middle. Fold it over to the right.



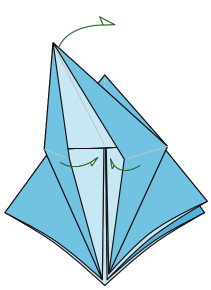
6. Fold one flap (two layers) so that the raw edge aligns with the center line; press firmly and unfold. Repeat on the right.



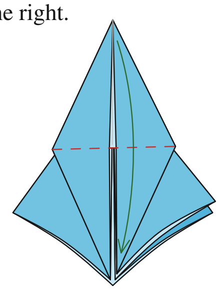
7. Fold the top flap down along a line connecting the points where the creases you just made hit the edges and unfold.



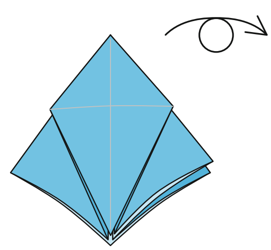
8. Lift up the bottom corner and push the sides in so that they meet along the center line.



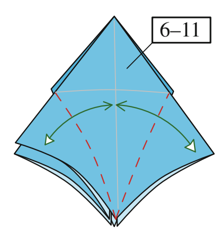
9. In progress. Flatten so that all layers lie together.



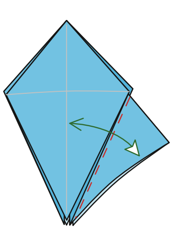
10. Fold the flap down.



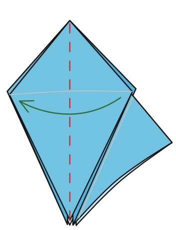
11. Turn the paper over from side to side.



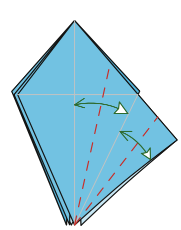
12. Repeat steps 6–11 on this side.



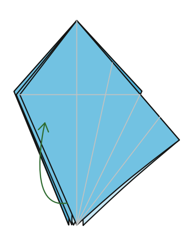
13. Fold and unfold along a crease aligned with existing edges.



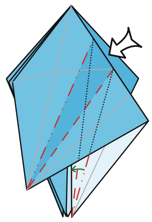
14. Fold one flap to the left.



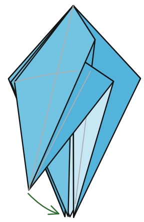
15. Bisect two angles on the wide flap on the right.



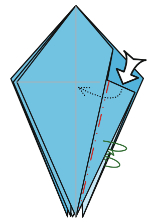
16. Lift up the near flap a bit in preparation for the next step.



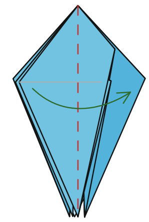
17. Using the existing creases, invert the shaded region so that the paper zigs in and out. Look at the next figure to see the result.



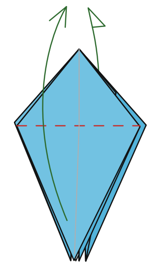
18. Flatten.



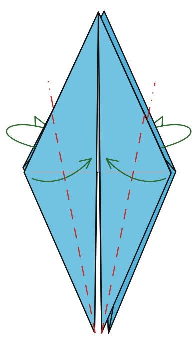
19. Invert the corner and tuck the edges inside.



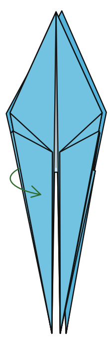
20. Fold one flap to the right.



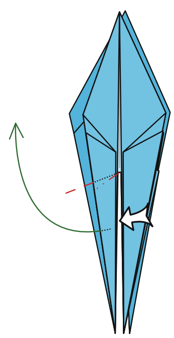
21. Fold one layer up in front and one layer up behind.



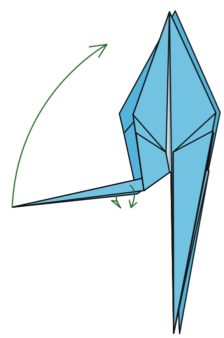
22. Fold two edges to the center line in front and two to the center line behind.



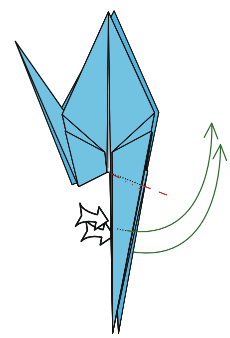
23. Open out the single point on the left side slightly.



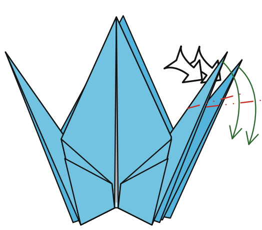
24. Push on the central edge of the flap so that it turns inside out between its upper edges.



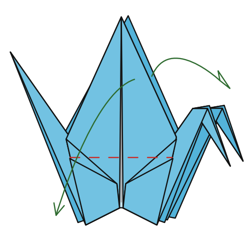
25. Swing the point up almost all the way to the top and fold its edges together. Flatten into the new position.



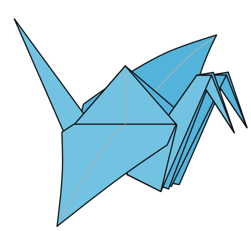
26. Repeat steps 24–25 on both of the points on the right.



27. Reverse the tips of both points so that they point downward, similarly to what you did in steps 24–25.



28. Fold one wing down in front and the other down behind.



29. The finished Hyperbolic Crane.