On the Magnus Function

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1 Introduction

There are uncountably many integer functions $M : Z \longrightarrow Z$ having the properties

$$M(0) = 0, M(M(x)) = x, M(x) = M(M(x-1)-1)-1.$$

Using this B. Neumann [*cf.* M2] showed the existence of uncountably many maximal non-parabolic subgroups of the modular group $PSL_2(Z)$.

In his article, [M1], Magnus inquires about the existence of functions $M : Z[1/2] \longrightarrow Z[1/2]$, on the ring of rational numbers with denominator a power of 2, the dyadic fractions $Z[1/2]$, which have the 'Neumann' properties displayed above and the additional properties,

$$M(2) = 2, M(x/2) = 2M(x).$$

Magnus wonders if there are only countably many maximal non-parabolic subgroups of the group $PSL_2(Z[1/2])$.

If such a Magnus function exists, it is determined by these displayed conditions and the values at the odd integers. Assuming existence of a Magnus function, M, it follows that $M(2) = \frac{1}{2}M(1)$ so that $M(1) = 4$, and also $M(-1) = 5,\ M(3) = 3,\ M(7) = 7,\ M(9) = 9$. Moreover it seems that M is odd integer valued for odd integers $\neq 1$; we call this property (O). Moreover, it appears that the function is uniquely determined from the conditions displayed above. We shall investigate this 'function'; some surprising relations

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hold *mod* 3, *mod* 7, and *mod* 9. The proof of the existence of M, the congruence conditions described below, and their distributions *mod* 2^n are unsolved problems for the interested reader.

2 Formulae and Congruences

One might look for a formula for M based on the binary represention of the dyadic fraction, or even to extend M to a function on the 2-adic numbers. For example the formulas

$$M(1+2^i x) = 2^i M(M(x) - 2^i) - 1,$$

$$M(2^i x - 1) = 2^i M(M(x) + 2^i) + 1,$$

follow easily from the axioms, evaluating at $\frac{x}{2^i}$ and simplifying. The first can be evaluated at $x = 1$ to give

$$M(1+2^i) = 2^{i-2} M(1 - 2^{i-2}) - 1$$

or $x = -1$ to give

$$M(1-2^i) = 2^i M(5 - 2^i) - 1.$$

The second can be evaluated at $x = 1$ to give

$$M(2^i - 1) = 2^{i-2} M(1 + 2^{i-2}) + 1$$

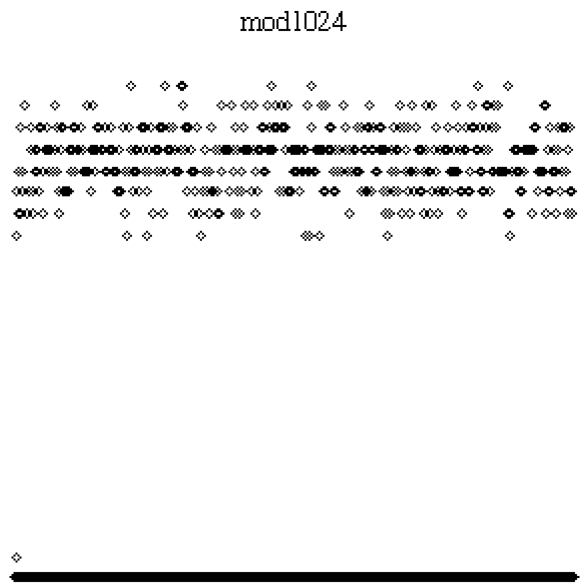
and at $x = -1$ yields

$$M(-2^i - 1) = 2^i M(5 + 2^i) + 1.$$

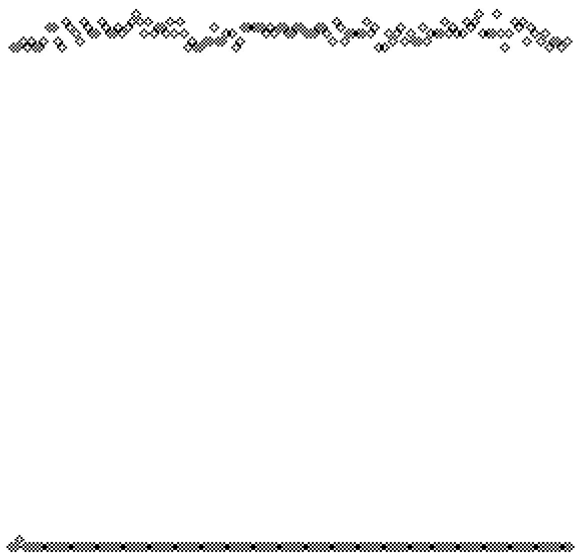
These formulae suggest that property (O) is valid and that there are congruential relations *mod* 2^n.

We can use these formulae to calculate $M(-15)$ easily as follows: $M(-15) = 16M(M(-1)-16)-1, M(-1) = 2M(3)-1 = 5$, and $M(-11) = 4M(M(-3)-4) - 1, M(-3) = 4M(1) - 1 = 15, M(11) = 4M(M(3) + 4) + 1 = 29$, so $M(-11) = 115$ and $M(-15) = 1839$.

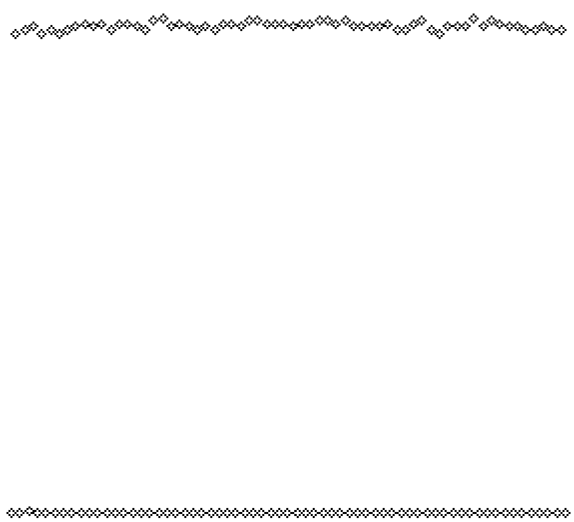
If we plot the histogram count in each of the congruence classes mod 1024, mod 256 and mod 128 of the values of M at odd $x, |x| < 10^4$ we obtain the following pictures. The bold line at the bottom of each picture denotes that there are no even residue classes, except $x = 1$. Essentially the odd congruence classes are evenly distributed $mod\ 2^n$ as $x \to \infty$.



mod256



mod128



3 Fixed Points and Congruences

Proposition: *If M has property (O) then any fixed point is one of 0,2,3,7,9.*

proof: First, if $0 \neq x \in Z[1/2]$ is not an odd integer and a fixed point, then $x = 2^v y, v \neq 0, y$ an odd integer, and $2^v y = x = M(x) = 2^{-v} M(y)$; thus $M(y)$ is not an odd integer; hence $M(y) = 4, y = 1$ so $x = 2$.

Next suppose that x is an odd integer, $M(x) = x$ then also $M(x+1) = M(x-1) - 1$; $x+1, x-1$ are even integers, $x - 1 = 2^v y, x + 1 = 2^w z, y, z$ odd integers; so applying this relation of the function M and the property (O) we have integers, a, b, so that $M(x-1) = \frac{a}{2^v}, M(x+1) = \frac{b}{2^w}$,

$$\frac{a}{2^v} - 1 = \frac{b}{2^w}.$$

For the first case suppose, $y, z \neq 1$, so that a, b are odd integers; since $x \neq \pm 1$, then $v, w \neq 0$, and from the displayed equation $v = w$. Hence $2 = x+1-(x-1) = 2^v(z-y)$, so that $v = 1$ and $z-y = 1$, which is impossible. For the second case, $z = 1$, then $M(x+1) = 2^{2-w} = M(x-1) - 1 = \frac{1}{2}M(2^{v-1} - 1) - 1$. From property (O), if $2^{v-1} - 1 \neq 1$ it follows that $w = 3$ and $x = 7$. Otherwise, $2^{v-1} - 1 = 1$ and $x = 3$. Finally, if $y = 1$ then $2^{2-v} - 1 = \frac{1}{2}M(2^{w-1} + 1)$; if $w \neq 1$ then from property (O), $v = 3$, and $x = 9$; if $w = 1$ the equation $2^{2-v} - 1 = 2$ has no solution.

We noticed some remarkable congruences at the fixed points. Besides the property (O), as far as we have calculated, the values at odd integers, $2x-1$, modulo various integers give the following values repeated cyclically.

$$M(2x-1) \begin{cases} mod\ 3 \\ mod\ 4 \\ mod\ 7 \\ mod\ 8 \\ mod\ 9 \end{cases} = \begin{cases} 1, 0, 2 \\ 1, 3 \qquad |x| > 6 \\ 4, 3, 6, 0, 2, 1, 5 \\ 5, 3, 1, 7 \quad |x| > 9 \\ 4, 3, 8, 7, 0, 2, 1, 6, 5 \end{cases}$$

Thus

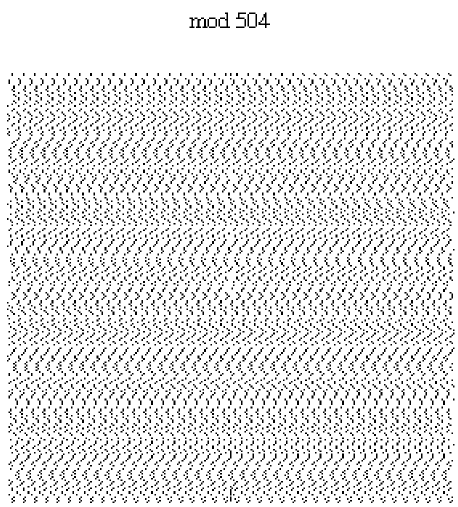
$$M(0 mod\ 3) = 0\ mod\ 3,$$

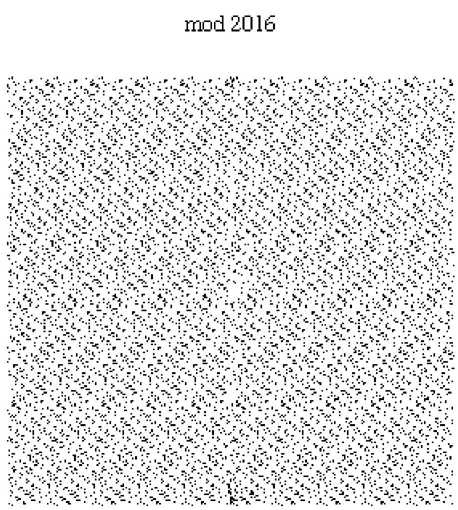
$$M(0 mod\ 7) = 0\ mod\ 7,$$

$$M(0 mod\ 9) = 0\ mod\ 9,$$

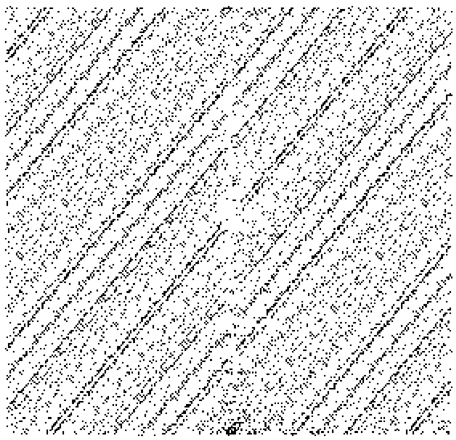
correspond to the fixed points at 3, 7, 9 described above.

These divisibilty properties for odd integers, x, mod 3, 4, 7, 8, 9 are visible in the graph of M mod 504 for $|x| \leq 10^4$. The distribution histogram for residue classes mod 504 is virtually bimodal.





mod 16128



mod 32256

References

[M1] Magnus, Wilhelm, *Rational Representations of Fuchsian Groups and Non-Parabolic Subgroups of the Modular Group*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl., 1-11, 1973.

[M2] Magnus, Wilhelm, **Noneuclidean Tessalations and Their Groups**, Academic Press, New York, 1974.

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Putative Values of M		
x	$M(x)$	$M(-x)$
1	4	5
3	3	15
5	-1	37
7	7	119
9	9	153
11	29	115
13	19	53
15	-3	1839
17	59	2353
19	13	3803
21	147	4893
23	233	55
25	295	1865
27	117	-45
29	11	565
31	73	-1441
33	951	-159
35	77	2443
37	-5	-211
39	57	87
41	1223	9433
43	3805	14915
45	1179	-27
47	209	79
49	847	1169
51	933	3387
53	-13	157
55	-23	29831
57	39	243465
59	17	1250803
61	475	-13291
63	945	80051391
65	29423	-5951
67	3781	2411
69	-93	1677