# Rationals and the Modular Group

**Roger C. Alperin**

The modular group $\mathscr{M}$ is the quotient group $PSL_2(\mathbf{Z}) = SL_2(\mathbf{Z})/\{\pm I\}$ of $SL_2(\mathbf{Z})$, the group of $2 \times 2$ integer matrices of determinant 1. In **[1]** we gave an elementary proof that $\mathscr{M}$ has the structure of a free product of a cyclic group of order 2 generated by the image of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and a cyclic group of order 3 generated by the image of $B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

The free product structure provides a description of the non-trivial elements of $\mathscr{M}$ as unique strings of $A$'s and $B$'s with the property that there are no two consecutive $A$'s and no three consecutive $B$'s; we refer to these as *reduced strings*. We explained this free product structure in terms of the action of the modular group on the irrationals. In this note we describe the action on the rationals; this can be viewed as a way of describing the inverse of the Euclidean algorithm.

The group $SL_2(\mathbf{Z})$ acts via linear transformations on $\mathbf{R}^2$ as column vectors and this gives an action of $\mathscr{M}$ via linear fractional transformations on the projective line $P^1(\mathbf{R})$, the real numbers together with $\infty$. We may also view $P^1(\mathbf{R})$ as the slopes of non-zero vectors, that is, the equivalence classes of $\mathbf{R}^2 - \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ induced by non-zero scalar multiplication; the equivalence class of the vector $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is denoted $\infty$, the equivalence class of the vector $\begin{pmatrix} p \\ q \end{pmatrix}$, $q \neq 0$ is the same as that of $\begin{pmatrix} p/q \\ 1 \end{pmatrix}$ and corresponds to the real number $z = p/q$. For the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $SL_2(\mathbf{Z})$ the induced action on $P^1(\mathbf{R})$ is given by

$$z \longrightarrow \frac{az+b}{cz+d}.$$

The induced action for the generating elements is given as

$$A\colon z \longrightarrow \frac{-1}{z}, \quad B\colon z \longrightarrow \frac{-1}{z+1}, \quad B^2\colon z \longrightarrow -1 - \frac{1}{z}.$$

The orbit of $e$ is easily seen to be in correspondence with the set of all first columns of matrices from $SL_2(\mathbf{Z})$. Thus the orbit of $\infty$ is in 1-1 correspondence with the projective line $P^1(\mathbf{Q})$, consisting of the set $\mathbf{Q}$ of all reduced fractions together with $\infty$; from elementary group theory this is in 1-1 correspondence with the set of left cosets of the stabilizer $\mathscr{N}$ of $e$, which is the image of the subgroup generated by $AB$ in $\mathscr{M}$.

Using the free product description of $\mathscr{M}$ we can also describe the set of coset representatives as reduced strings of $A$'s and $B$'s . First, a non-trivial coset representative cannot end in $AB$ or its inverse $B^2A$; therefore if it ends in $A$ it is either $A$ or of the form $ZBA$ with $Z$ ending in $A$ or trivial; if it ends in $B$ it is either $B$ or $ZB^2$ with $Z$ ending in $A$ or trivial. Thus, as a first pass, the set of coset representatives is the set $\mathscr{R} = \{I\} \cup \{A\} \cup \{B\} \cup \{BA\} \cup \{B^2\} \cup \{ZBA | Z$ any string ending in $A\} \cup \{ZB^2 | Z$ any string ending in $A\}$. Next, to determine the distinct coset representatives we just observe that the free product description gives a unique expression for the elements. The coset equivalence relation $X \cong Y$ on reduced strings $X, Y \in \mathscr{M}$ is $Y = X(AB)^n$ or $Y = X(B^2A)^n$ for some non-negative integer $n$. We see easily that $A \cong B \bmod \mathscr{N}$ and hence also $XA \cong XB \bmod \mathscr{N}$ for any string $X$, and hence if this is reduced, $X \neq I$ must end in $B$. Thus we can simplify the description of the distinct coset representatives to $\mathscr{R} = \{I\} \cup \{B\} \cup \{B^2\} \cup \{ZB^2 | Z \text{ any string ending in } A\}$. It is easy to see that no two of these reduced strings are equivalent; for example, for $Z \neq W$, both ending in $A$, then $ZB^2 = WB^2(B^2A)^n$ and $ZB^2 = WB^2(AB)^n$ are impossible. Thus the coset representatives of $\mathscr{M}/\mathscr{N}$ are the distinct strings $\mathscr{R} = \{I\} \cup \{B\} \cup \{B^2\} \cup \{ZB^2 | Z$ any string ending in $A\}$.

We can also describe this set $\mathscr{R}$ as the union of $\mathscr{R}_m$ defined inductively as

$$\begin{aligned} \mathscr{R}_0 &= \{I, B\}, \quad \mathscr{N}_0 = \{B^2\} \\ \mathscr{P}_m &= \{A\}\mathscr{N}_m, \quad \mathscr{N}_{m+1} = \{B^2, B\}\mathscr{P}_m \\ \mathscr{R}_{m+1} &= \mathscr{R}_m \cup \mathscr{P}_m \cup \mathscr{N}_m \end{aligned} \tag{1}$$

It is easy to see that $\mathscr{R}_m$ has $2^{m+1}$ elements and $\mathscr{P}_m$, $\mathscr{N}_m$ each have $2^m$ elements. We can rewrite (1) as

$$\mathscr{P}_{m+1} = \{AB, AB^2\}\mathscr{P}_m, \quad \mathscr{N}_{m+1} = \{B^2A, BA\}\mathscr{N}_m. \tag{2}$$

Simplifying (2) we obtain the following result. $\mathscr{F}\mathscr{S}(x, y)$ denotes the free semigroup with the generators, $x, y$.

**Proposition.** *$P^1(\mathbf{Q})$ is in 1-1 correspondence with* $\mathscr{R} = \{I\} \cup \{B\} \cup \mathscr{F}\mathscr{S}(AB, AB^2) \cdot AB^2 \cup \mathscr{F}\mathscr{S}(B^2A, BA) \cdot B^2$.

Observing, that $I\infty = \infty$, $B\infty = 0$, $B^2\infty = -1$ and $AB^2\infty = 1$, and $P^1(\mathbf{Q}) - \{\infty, 0\}$ is the positive and negative rationals, we have the following

**Corollary.** *The set of positve rationals is the orbit of the free semigroup generated by $AB$ and $AB^2$ on $z = 1$. The set of negative rationals is the orbit of the free semigroup generated by $B^2A$ and $BA$ on $z = -1$.*

These upper $U = AB$, $U^- = B^2A$ and lower $L = AB^2$, $L^- = BA$ triangular matrix actions corresponding to these semigroup generators are

$$U: z \longrightarrow z + 1, \quad L: z \longrightarrow \frac{z}{z+1},$$

$$U^-: z \longrightarrow z - 1, \; L^-: z \longrightarrow \frac{z}{1-z}.$$

Every positive rational is uniquely expressible in terms of semigroup generators as an element of the orbit of 1. Alternatively, starting from a reduced positive rational $z = \frac{p}{q}$ we can apply a greedy or Euclidean recursion to obtain a finite sequence that stabilizes at 1;

$$\frac{p}{q} \longrightarrow \begin{cases} \dfrac{(p-q)}{q} & \text{if } p > q \\ \dfrac{p}{(q-p)} & \text{if } p < q \\ 1 & \text{if } p = q = 1 \end{cases}$$

Here we apply $U^-$ or $L^-$ depending on whether or not $z > 1$ or $z < 1$. For example, the sequence

$$\frac{34}{55}, \frac{34}{21}, \frac{13}{21}, \frac{13}{8}, \frac{5}{8}, \frac{5}{3}, \frac{2}{3}, \frac{2}{1}, 1$$

corresponds to $(U^- L^-)^4(\frac{34}{55}) = 1$. Since we have shown that the set of positive rationals can be described by a free semigroup, this means that $(LU)^4 L$ is *the* coset representative in $\mathscr{R}_9$ corresponding to 34/55 as described in the Corollary.

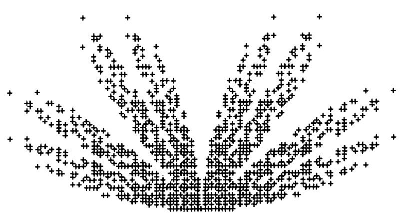

**Figure 1.** $\mathscr{R}_9$.

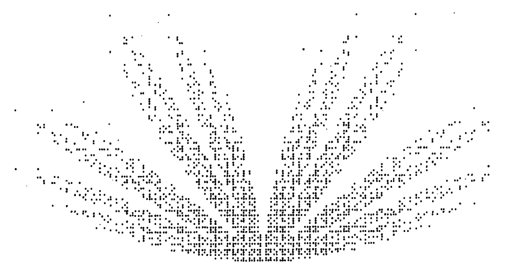

**Figure 2.** $\mathscr{R}_{10}$.

Finally, we consider the matrix action of the distinct non-trivial coset representatives in $\mathscr{R}$ on the column $\binom{1}{0}$ and the plots of these images in $\mathbf{R}^2$. These images are just the points with relatively prime coordinates in the upper half-plane. We obtain the fascinating plant-like structures in Figures 1 and 2. For example, the distant points from the root $\binom{0}{1}$ are the 'Fibonacci points' $\binom{\pm 34}{55}$, $\binom{\pm 55}{34}$ in $\mathscr{R}_9$.

REFERENCE

1. Roger C. Alperin, $PSL_2(\mathbf{Z}) = \mathbf{Z}_2 * \mathbf{Z}_3$ *Amer. Math. Monthly* **100** (1993) 385–386.

*alperin@mathcs.sjsu.edu*