SOMOS MEETS FIBONNACI

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Abstract. We describe certain elementary non-linear sequences which are integer valued and characterize the integral sequences for the special example $x_{n+1}x_{n-1} = x_n^2 + 1$; this is related to the alternate terms of the Fibonacci sequence.

1. Introduction

We are interested in the sequences generated by the non-linear equation

$$x_{n+1}x_{n-1} = x_n^2 + A$$

with constant $A \neq 0$ with initial values specified for x_1, x_2. Specifically we want to know for a given A which integer values of x_1 and x_2 will give a sequence consisting only of integers. This is a simplified version of a Somos sequence; it is known that the sequences generated by this equation generally have denominators of the form $x_1^{n-1}x_2^{n-2}$ [1].

Here are two simple examples: for the $A = 1$ sequence $x_1 = 1, x_2 = 1$, then the successive terms are $2, 5, 13, \ldots$, the alternate terms of the Fibonacci sequence; for the $A = -1$ sequence $x_1 = 1, x_2 = 2$, gives the sequence of positive integers.

2. A-sequence

Surprisingly the sequences generated by $x_{n+1}x_{n-1} = x_n^2 + A$ are linearly recursive. We denote by $\Sigma = \Sigma_A(x_1, x_2)$ a sequence determined by the formulas given above; we refer to this as an A-sequence.

Theorem 2.1. *Let $\mu = \frac{x_2^2 + x_1^2 + A}{x_1 x_2}$. The sequence satisfies*

$$x_{n+1} = \mu x_n - x_{n-1}.$$

Proof. We show that $\frac{x_{n+1} + x_{n-1}}{x_n}$ is constant and equal to μ. Certainly this equality is valid for $n = 2$: $\frac{x_3 + x_1}{x_2} = \mu$. Now

$$\frac{x_{n+2} + x_n}{x_{n+1}} = \frac{x_{n+1}^2 + x_n^2 + A}{x_{n+1}x_n}.$$

Inductively then,

$$\begin{aligned}\mu - \frac{x_{n+2}+x_n}{x_{n+1}} &= \frac{x_{n+1}+x_{n-1}}{x_n} - \frac{x_{n+1}^2+x_n^2+A}{x_{n+1}x_n}\\ &= \frac{x_{n+1}^2+x_{n-1}x_{n+1}-x_{n+1}^2-x_n^2-A}{x_{n+1}x_n}\\ &= \frac{x_{n-1}x_{n+1}-x_n^2-A}{x_{n+1}x_n}\\ &= 0\end{aligned}$$

□

In a similar way one can show that the sequences

$$x_{n+1}x_{n-1} = x_n^2 + Bx_n + A$$

are linearly recursive of degree 3 with characteristic equation $X^3 - \mu X^2 + \mu X - 1$ where $\mu = \frac{x_1^2+x_2^2+x_1x_2+B(x_1+x_2)+A}{x_1x_2}$. Note that 1 is a root of the characteristic equation.

Also the sequence

$$x_{n+1}x_{n-2} = x_n x_{n+1} + A$$

satisfies the linear recurrence $x_{n+1} = \mu x_{n-1} + x_{n-3}$ and $\mu = \frac{x_1(x_0^2+x_2^2)+A(x_0+x_2)}{x_0x_1x_2}$.

3. *A*-Integer Sequences

3.1. **Examples.** Here is a method to generate integral sequences. Let x_1, x_2 determine μ as before, say $x_1, x_2 \in \{r, s\} \subset \mathbb{Z}$ with $\frac{s+1}{r} \in \mathbb{Z}$, and $A = s - r^2$ then $\mu = \frac{r^2+s^2+s-r^2}{rs} = \frac{s+1}{r}$. Certainly r, s have no common factor and the sequence $\Sigma_A(r, s)$ consists of integers.

For example with $x_1 = r = 1, x_2 = s, \mu = s+1, A = s-1$, we obtain an integral sequence for any integer value of A.

3.2. **Criteria.** If $\mu \in \mathbb{Z}$ with initial integral x_1, x_2 then all the x_i are integers.

Let d denote the denominator of μ. Conversely, if the sequence is integral then $x_n\mu \in \mathbb{Z}$ for $n \geq 2$.

Lemma 3.1. *If the A-sequence is integral then* $d^{n-1}|x_k$ *for all* $k \geq n \geq 1$.

Proof. By induction, the case $n = 0$ being vacuously true. Consider the matrix $M = \begin{pmatrix}\mu & -1\\ 1 & 0\end{pmatrix}$. Then $M^n\begin{pmatrix}x_2\\ x_1\end{pmatrix} = \begin{pmatrix}x_{n+1}\\ x_n\end{pmatrix}$. Inductively the bottom row of M^{-n} has $2-2$ entry which is a polynomial $p(\mu)$ of degree $n-1$ in μ and the $2-1$ entry is a polynomial $q(\mu)$ of degree $n-2$ in μ. Hence $qx_{n+1} + px_n = x_1$ so $d^{n-1}x_n \in \mathbb{Z}$; since this is true for all n the conclusion of the Lemma follows also for $k \geq n$. □

Proposition 3.2. *If the A-sequence is integral then* $\mu \in \mathbb{Z}$.

Proof. From the Lemma the denominator d of μ has the property that $d^n | x_{n+1}$ for all $n \geq 0$. Using $x_{n+1}x_{n-1} = x_n^2 + A$ we get that $d^{2n-2}|A$ for all $n \geq 1$; hence $d = 1$. $\square$

3.3. **Pell's Equation.** Suppose that $x_1 = a, \mu \in \mathbb{Z}$, then using the formula for μ we have an integer equation

$$x^2 - a\mu x + A + a^2 = 0$$

which will have an integer solution $x = x_2 \in \{\frac{a\mu}{2} \pm \frac{\sqrt{(a\mu)^2 - 4A - 4a^2}}{2}\}$ iff the discriminant is an integer square c^2 and $a\mu \pm c$ is even. Hence also we have an integer solutions $X = c, Y = a$ to the Pell's equation

$$X^2 - (\mu^2 - 4)Y^2 = -4A. \tag{3.1}$$

Theorem 3.3. *The A-sequence is integral iff there are integer solutions when c is even to: $X^2 - \frac{\mu^2-4}{4}Y^2 = -A$ when μ is even or $X^2 - (\mu^2 - 4)Y^2 = -A$ when μ is odd; or when c is odd then μ is odd and $X^2 - (\mu^2 - 4)Y^2 = -4A$ has a solution with X odd.*

Proof. With the notation as above, suppose first that c is even. If $\mu^2 - 4$ is odd then a is even so the equation reduces to $X^2 - (\mu^2 - 4)Y^2 = -A$. If $\mu^2 - 4$ is even then μ is also even and then the equation reduces to $X^2 - ((\frac{\mu}{2})^2 - 1)Y^2 = -A$.

If however c is odd then $\mu^2 - 4$ and a are both odd and the equation remains as $X^2 - (\mu^2 - 4)Y^2 = -4A$.

Conversely if we have solutions to the Pell's equation 3.1 above then we can make an A-sequence integral solution using the solution for $x_1 = Y$ or $x_1 = 2Y$ and then solve for x_2 using the quadratic formula given $\mu^2 - 4$ with known x_1, A. The equation is simply

$$x_2^2 + A + a^2 - \mu a x_2 = 0 \tag{3.2}$$

$\square$

4. $A = 1$

There may not be a unit of norm -1 or -4 in the associated ring for the Pell equation, $X^2 - rY^2 = -1$, $X^2 - rY^2 = -4$. The existence of the unit of norm -1 depends on whether or not the period of the continued fraction of $\sqrt{r}$ is odd [2].

If $r = \mu$ is odd and $r > 3$ then $\sqrt{r^2 - 4}$ has even period since $\sqrt{r^2 - 4} = (r - 1; \overline{1, \frac{r-3}{2}, 2, \frac{r-3}{2}, 1, 2r - 2})$. If $s = \frac{\mu}{2}$ is an integer then for $s \geq 2$, $\sqrt{s^2 - 1} = (s - 1; \overline{1, 2s - 2})$ has even period.

Theorem 4.1. *If $A = 1$ then the integral A-sequences exist iff $\mu = \pm 3$. Any integer solution to $X^2 - 5Y^2 = -1$ gives an integral A-sequence with $x_1 = Y$ and x_2 a solution to the quadratic equation $x_2^2 - \mu x_1 x_2 + 1 + x_1^2 = 0$.*

Proof. We have shown above there are no solutions to the Pell's equation $X^2 - (\mu^2 - 4)Y^2 = -1$, $\mu \neq \pm 3$. Also we have shown above there are no solutions to $X^2 - \frac{\mu^2-4}{4}Y^2 = -1$ for μ even and $\frac{\mu}{2} \geq 2$. For the last case we consider solutions to the Pell's equation

$$X^2 - (\mu^2 - 4)Y^2 = -4$$

with $X = c$ odd; hence μ is odd and $Y = a$ is also odd. We may assume that $\mu^2 - 4$ is square-free since any square factor can be absorbed into the solution for Y. In this situation using the congruence mod 4 we see that the Pell's equation has no solution if $\mu^2 - 4 \equiv 3 \mod 4$.

Suppose then that $D = \mu^2 - 4 \equiv 1 \mod 4$. The algebraic integers $\mathbb{Z}_D$ in the field $\mathbb{Q}(\sqrt{D})$ properly contains the ring $\mathbb{Z}[\sqrt{D}]$. If the fundamental unit of $\mathbb{Z}_D$ does not lie in $\mathbb{Z}[\sqrt{D}]$ then we get the desired solution to the Pell's equation. Conversely if we have the desired solution X, Y odd then we get a unit in $\mathbb{Z}_D$ which does not lie in $\mathbb{Z}[\sqrt{D}]$. However the cube of this unit lies in the ring $\mathbb{Z}[\sqrt{D}]$ which means that there is a solution to the Pell's equation $x^2 - (\mu^2 - 4)y^2 = -1$; but this impossible since the period is even. (Note that $\mu^2 - 4 \equiv 1 \mod 8$ is impossible since there is no solution to $x^2 = 5 \mod 8$. Also $\mu^2 - 4 \equiv 5 \mod 8$ is used to show the cube of a unit in the larger ring lies in the smaller ring.) ☐

If we also reverse the sequence to include $x_n, n \leq 0$ then essentially there are just 4 sequences when $A = 1$ ignoring the exact starting place.

The solutions for $r = \mu = \pm 3$ correspond to *odd* powers of the fundamental unit $\alpha = \frac{1+\sqrt{5}}{2}$ or its inverse $\alpha^{-1} = \frac{-1+\sqrt{5}}{2}$ and are related to the alternate terms of the Fibonacci sequence.

Corollary 4.2. *The integral sequences for $A = 1$ have starting values x_1, x_2 which are consecutive terms in one of the four bi-infinite sequences listed here:*

$$\ldots, -89, 34, -13, 5, -2, 1, -1, 2, -5, 13, -34, 89, \ldots,$$

$$\ldots, 89, -34, 13, -5, 2, -1, 1, -2, 5, -13, 34, -89, \ldots,$$

$$\ldots, -89, -34, -13, -5, -2, -1, -1, -2, -5, -13, -34, -89, \ldots,$$

$$\ldots, 89, 34, 13, 5, 2, 1, 1, 2, 5, 13, 34, 89, \ldots.$$

Proof. From the Theorem we need to consider $\mu = \pm 3$ and the solutions to $X^2 - 5Y^2 = -4$. The solutions are the odd powers of $\pm\alpha, \pm\alpha^{-1}$ which give the sequences listed above. ☐

References

[1] S. Fomin, A. Zelevinsky, The Laurent Phenomenon, Advances in App. Math., 28, 119-144, (2002).

[2] W. Sierpinski, *Elementary Theory of Numbers*, Warsaw, 1964.

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