Folding the Regular Pentagon Using Bisections and Perpendiculars

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There are numerous origami constructions for the pentagon, see for example [**3**]. Dureisseix [**2**] provided constructions for optimal polygons inscribed in a square using origami folds. The optimal pentagon has four vertices on the four edges of the square and one vertex on the diagonal so that the pentagon has that diagonal as an axis of symmetry.

However, the construction is known to be a 'Pythagorean' construction [**1**], that is, we can achieve it using only the origami bisection of angles or the construction of a perpendicular to a line through a given point. We accomplish this for a maximal inscribed pentagon in the square by modifying Dureisseix' construction (notably in his second step).

Suppose the side length of our square $OPQR$ is one. Fold the hoizontal midline MN of the square parallel to RO. From the vertex P construct the diagonal PM and the angle bisector PA of $\angle MPO$ with A on the edge RO. Let $\theta = \angle APO$, $x = \tan(\theta) = OA$, then $2\theta = \angle MPO$ and $\tan(2\theta) = 2 = \frac{MN}{PN}$; hence x satisfies $\frac{2x}{1-x^2} = 2$ so $x = \frac{\sqrt{5}-1}{2}$.

Construct A_1 on the side OP by folding A across the diagonal at O so that $OA_1 = OA$. Construct A_2 as midpoint of OA. Fold the line A_2N to YN on the midline MN by using a bisection of MNA_2. The length of YN is $y = \frac{1}{2}\sqrt{1+x^2}$ so $y = \frac{\sqrt{10-2\sqrt{5}}}{4}$.

Now we determine the product xy using similar triangles: $2y = \frac{YN}{\frac{1}{2}} = \frac{B_1A_1}{A_1O} = \frac{B_1A_1}{x}$. Since $2xy = B_1A_1$ we let B_2 be the midpoint of B_1A_1; moving it to the midline gives BN with length xy.

Construct OC as the angle quartisection of $\angle BOP$. Now C_1 is constructed on diagonal OQ so that $OC = OC_1$.

In the diagram below we use use solid line for a fold between two points, long-dash for a bisection and short-dash line for a perpendicular or parallel.

We now fold the pentagon as follows:

(1) Construct vertex E by dropping a perpendicular from C_1 to QP;
(2) Construct vertex F on RO by constucting EF parallel line to OC.
(3) Construct vertex G on RQ by folding F across PR.
(4) Construct vertex H on OP by folding E across PR.
(5) Construct vertex I on PR by folding F across the angle bisector of $\angle GHE$.

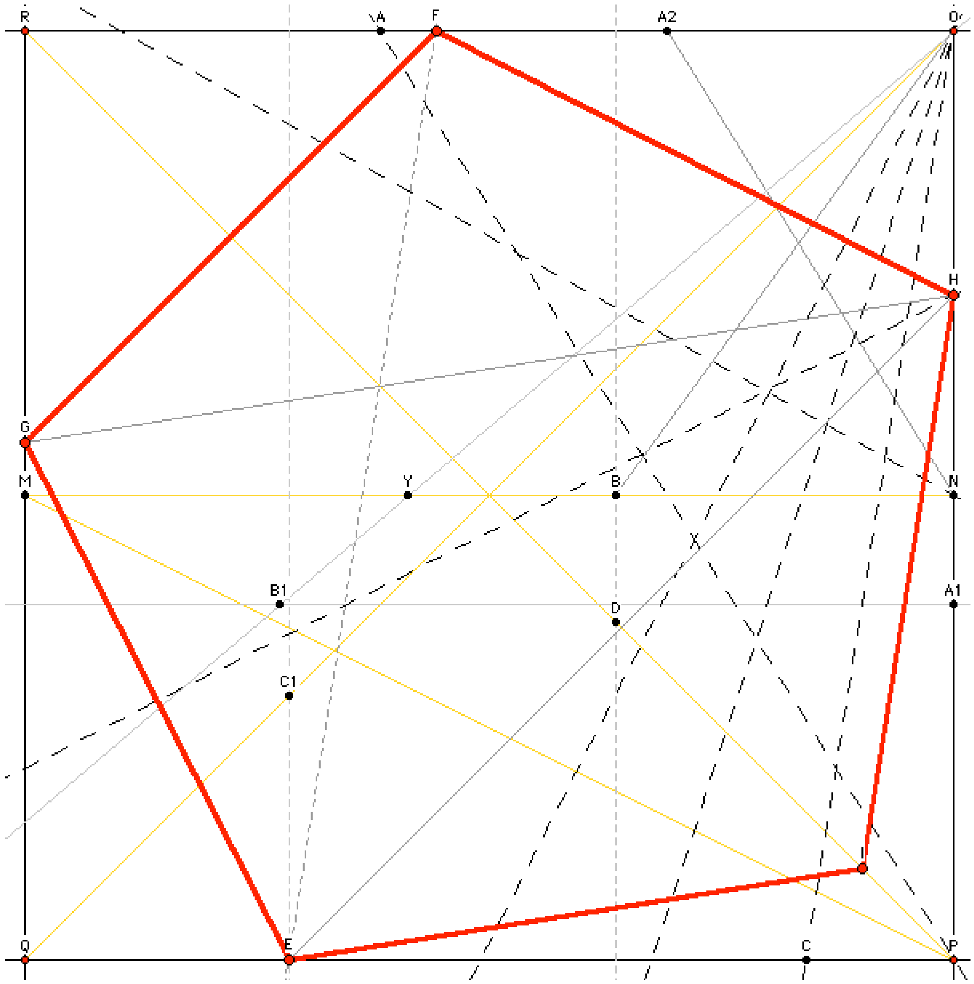

FIGURE 1. Fold lines for the maximal regular pentagon inscribed in a square

The proof that this is a regular pentagon is now easy. By construction $GE = FH = HI$ and $HI = IE$ by symmetry. Also $GH = EH$ and $\angle EHI = \angle GHF$ by construction and consequently by similar triangles $GF = EI$.

References

[1] Roger C. Alperin, *A Mathematical Theory of Origami Constructions and Numbers*, New York J. Math., 6, (2000) 119-133 *http://nyjm.albany.edu*

[2] David Dureisseix, *Folding Optimal Polygons from Squares*, Math. Mag., 79(4), 2004, 272-279.

[3] Roberto Morassi, *The Elusive Pentagon*, Proceedings of Origami Science and Technology, ed. H. Huzita, 1990.

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