A Characterization of Finite Commutative Rings

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We give a partial converse to the well-known result: 'If R is finite commutative ring with identity then every element is a unit or a zero divisor'. An important partial converse of this which we use here is that: 'A field which is finitely generated as a ring is finite'. We hope students of commutative algebra may find our proof interesting and enlightening.

Theorem 1 *Let R be a commutative ring with unity which is finitely generated as a ring. If evey element of R is either a unit or a zero divisor then R is finite.*

Proof: Let $Z(R)$ be the zero divisors of the ring R. Since a finitely generated ring is a quotient of a polynomial ring over $\mathbb{Z}$ in finitely many indeterminates, it is Noetherian. The primary decomposition theorem gives $(0) = \bigcap_{k=1}^{n} I_k$ for primary ideals I_k; $rad(I_k) = P_k$. As a consequence $Z(R) = \bigcup_{i=1}^{n} P_i$, [1, Prop. 4.7]. It follows immediately from the hypothesis that any proper ideal I of R can consist only of zero divisiors. Thus $I \subset \bigcup_{i=1}^{n} P_i$; so using [1, Prop. 1.11] $I \subset P_i$ for some i. In particular all maximal ideals are among the P_i, $1 \le i \le n$. Hence there are finitely many maximal ideals of R.

We now show that all prime ideals of R are maximal. One of the equivalent properties for a Jacobson ring is that every prime ideal is an intersection of maximal ideals. The ring $\mathbb{Z}$ is Jacobson follows from $(0) = \bigcap_{p\ prime}(p)$. As shown in [1, p. 71, No. 24], a finitely generated algebra over a Jacobson ring is also Jacobson. Hence R is also a Jacobson ring. Since there are finitely many maximal ideals any prime ideal P is an intersection of *finitely* many maximal ideals $P = \bigcap_{i=1}^{n} M_i$ so by [1, Prop. 1.11] $P = M_k$ for some k. Hence prime ideals of R are maximal.

Now we use Artinian properties, since Noetherian and Krull dim(R)=0 (i.e., non-zero prime ideals are maximal) is equivalent to Artinian [1, Chap. 8]. By the structure theorem for Artin rings [1, Theorem 8.7], R is a product of Artin local rings, where the maximal ideal M is nilpotent [1, Theorem 8.6]; say $M^n = \{0\}$. Now we have a filtration $\{0\} \subset M^{n-1} \subset \ldots M^{i+1} \subset M^i \cdots \subset M \subset R$ and $M^i/M^{i+1} \cong R/M$. Since R/M is a field finitely generated as a ring it is finite [1, p. 71, No. 25]. Since R/M is a finite field then also the layers in the filtration above are all finite and hence R is also finite. $\square$

References

[1] M. F. Atiyah and I. G. MacDonald, **Introduction to Commutative Algebra**, Addison-Wesley, 1969.

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