

Other variants of the secretary problem were studied later for partial orders (complete binary tree [7], general partial order [5, 8]), for graphs and digraphs [4], and threshold stopping times [3].

The game described in this paper could be generalized to allow  $p$  different types of elements, with  $p > 2$ . If the objective is to “choose rarity” by stopping on an element in the smallest set, an optimal strategy seems to be analogous to the one described in the paper, but finding probabilities of winning is more challenging.

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**Summary** You enter an online game, hoping to win a prize. You know that thirteen envelopes of two colors will appear one by one on your screen, and you know that there will be five of one color and eight of the other. Each of the five envelopes, those with the “rare” color, contains \$100, while the others are empty. However, you do not know which color is which, and you can only select one envelope; once you click, the game is over. How should you play to maximize the probability of winning? We provide an optimal strategy for this game and a generalization.

## Golomb Rulers

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The Math Factor podcast posed the problem of finding the smallest number of inch marks on a 12 inch ruler so that one could still measure any integer length from 1 to 12 as a distance between marks. One needs only four additional marks besides 0 and 12; for example 1, 4, 7, 10 works, as you can check. This entertaining problem led to others during the next few minutes (you can listen at <http://mathfactor.uark.edu/2005/10>) and inspired us to look for generalizations. After several false starts

and numerous literature searches we uncovered the theory of *Golomb* and *minimal spanning rulers*, a generalization to the natural numbers and relations of an unsolved conjecture of Erdős and Turan.

We analyze our first problem, which led us to Golomb rulers. A property of the ruler of length 6 with marks at 0, 1, 4, 6 is that each of the lengths 1, 2, 3, 4, 5, and 6 can be measured and it can be done in only one way. *Can one choose marks on a ruler of length 12 so that each length from 1 to 12 can be measured in only one way?*

Golomb rulers are sets of integers (marks) with the property that if a distance can be measured using these marks then it can be done in a unique way.

DEFINITION 1. A set  $\mathcal{G}$  of integers

$$a_1 < a_2 < \cdots < a_{p-1} < a_p$$

is called a *Golomb ruler* if for every two distinct pairs of these integers, say  $a_i < a_j$  and  $a_m < a_n$ , we have  $a_j - a_i \neq a_n - a_m$ .

The *size* of  $\mathcal{G}$  is defined to be  $p$  (the number of marks in  $\mathcal{G}$ ) and is denoted  $\#\mathcal{G}$ . The *length* of  $\mathcal{G}$  is defined to be  $a_p - a_1$  (the largest distance that can be measured using the marks from  $\mathcal{G}$ ).

It is clear that we can translate these sets: if  $\mathcal{G} = \{a_1, a_2, \dots, a_p\}$  is a Golomb ruler then so is  $\{a_1 + b, a_2 + b, \dots, a_p + b\}$ . This makes the choice of  $a_1$  immaterial, so it will usually be taken to be 0. It is also clear that if  $\mathcal{G} = \{a_1, a_2, \dots, a_p\}$  is a Golomb ruler, then so is the reflection of  $\mathcal{G}$  around the midpoint  $(a_1 + a_p)/2$ . For example,  $\{0, 1, 4, 6\}$  is a Golomb ruler, as is  $\{0, 2, 5, 6\}$  obtained by reflecting the first ruler around the point 3. To simplify the statements of some of the theorems, a set  $\{a_1\}$  consisting of a single point is considered to be a Golomb ruler.

Golomb rulers have numerous applications. The best known is an application to radio astronomy. Radio telescopes (antennas) are placed in a linear array. For each pair of these antennas, the received signals are subtracted from each other and an inference can be then made as to the location of the source. These inferences can be made much more accurate if all the distances between the antennas are multiples of the same common length, and many such pairs with distinct distances between them are available and can be utilized. The problem maximizing the number of distinct distances between the pairs, while minimizing the number of the antennas and the length of the array, was first considered by Solomon W. Golomb [1, 2, 8, 10].

Other applications include assignments of channels in radio communications, X-ray crystallography, and self-orthogonal codes. Rankin [12] gives more information about these applications. There is also a wealth of information in various writings by Martin Gardner [5, 6, 7].

The Golomb ruler  $\{0, 1, 4, 6\}$  has the additional property that every integral distance between 1 and 6 can be measured. We call such a ruler *perfect*.

DEFINITION 2. A Golomb ruler  $\mathcal{G}$  of length  $N$  is called *perfect* if every integer  $d$ ,  $1 \leq d \leq N$ , can be expressed as  $d = a - a'$ , for some  $a, a' \in \mathcal{G}$ .

Since  $\mathcal{G}$  is a Golomb ruler, the representation of each  $d$  is unique. Unfortunately, there are very few perfect Golomb rulers.

THEOREM 1. (GOLOMB) *Together with their translations and reflections around the midpoint the only perfect Golomb rulers are  $\{0\}$ ,  $\{0, 1\}$ ,  $\{0, 1, 3\}$ , and  $\{0, 1, 4, 6\}$ .*

This theorem was proved by Golomb, but apparently he never published it. There are several places where the proof appears [4, 12], but they are not very easily accessible, so we present here a slight modification of the original argument.

*Proof.* If  $\mathcal{G}$  is a perfect Golomb ruler of size  $p$  and length  $N$ , then, since there are  $N$  distances to be measured and the number of distinct pairs of these points is  $\binom{p}{2}$ , we must have  $N = \frac{1}{2}p(p-1)$ , so  $N$  is a triangular number. The triangular numbers below 10 are 0, 1, 3, 6 corresponding to the rulers listed in the theorem.

Assume then that  $\mathcal{G}$  is a perfect Golomb ruler of length  $N > 9$  and seek a contradiction. Without loss of generality we may assume that  $a_1$ , the smallest number in  $\mathcal{G}$ , is equal to 0 and so the largest number is  $a_p = N$ . By hypothesis, every number  $1 \leq d \leq N$  is uniquely realizable as a difference of two marks in  $\mathcal{G}$ . Since  $N$  is realizable, 0 and  $N$  must belong to  $\mathcal{G}$ . Since  $N-1$  is realizable, either 1 or  $N-1$  belongs to  $\mathcal{G}$ . By reflecting  $\mathcal{G}$  around  $N/2$ , we may assume that  $1 \in \mathcal{G}$ . Next, since  $N > 3$ ,  $N-2$  must be realized. Since  $N-2 > 1$ ,  $\mathcal{G}$  must contain another point.

The possible pairs realizing  $N-2$  are  $\{2, N\}$ ,  $\{1, N-1\}$ ,  $\{0, N-2\}$ . The first two produce duplications:  $1-0 = 2-1$  and  $1-0 = N-(N-1)$ . The third is the only possibility, so  $\mathcal{G}$  contains  $N-2$  as well as 0, 1,  $N$ . The realized distances are 1, 2,  $N-3$ ,  $N-2$ ,  $N-1$ , and  $N$ .

Since  $N-4 \notin \{1, 2\}$  we need one of the pairs  $\{0, N-4\}$ ,  $\{1, N-3\}$ ,  $\{2, N-2\}$ ,  $\{3, N-1\}$ ,  $\{4, N\}$  to realize  $N-4$ . All but the last case produce distances already realized:  $(N-2) - (N-4) = N - (N-2)$ ,  $1-0 = (N-2) - (N-3)$ ,  $2-0 = N - (N-2)$ ,  $1-0 = N - (N-1)$ .

The last case passes inspection, so  $\mathcal{G}$  contains 0, 1, 4,  $N-2$ , and  $N$ . The distances that can be realized by  $\mathcal{G}$  are 1, 2, 3, 4,  $N-6$ ,  $N-4$ ,  $N-3$ ,  $N-2$ ,  $N-1$ , and  $N$ .

Finally, consider the distance  $N-5$ . Since  $N-5 \notin \{1, 2, 3, 4\}$  and  $N > 9$  this distance has not been realized. The possible pairs for realizing the distance  $N-5$  are  $\{0, N-5\}$ ,  $\{1, N-4\}$ ,  $\{2, N-3\}$ ,  $\{3, N-2\}$ ,  $\{4, N-1\}$ ,  $\{5, N\}$ . The reader may easily check that each of these case leads to a duplication. This contradiction shows that  $N < 9$  and the constructions above give the perfect rulers asserted by the theorem. ■

Since perfect Golomb rulers essentially do not exist, we seek “almost perfect” rulers. Roughly speaking, given a length  $N$ , we try to place as many points as possible in the interval  $[0, N]$  so that the resulting set forms a Golomb ruler. Alternatively, given the size  $p$  of the ruler (the number of marks), we try to construct a Golomb ruler of shortest possible length  $N$  with  $p$  points. Such rulers are called optimal.

**DEFINITION 3.** For every positive integer  $p$ , let  $G(p)$  be the shortest possible length of a Golomb ruler with  $p$  marks.

A Golomb ruler with  $p$  marks is called *optimal* if its length is  $G(p)$ . Dimitro-manolakis [4] discusses optimal Golomb rulers in detail. For example,  $G(6) = 17$ , and there are 4 optimal rulers of size 6 and length 17:  $\{0, 1, 4, 10, 12, 17\}$ ,  $\{0, 1, 4, 10, 15, 17\}$ ,  $\{0, 1, 8, 11, 13, 17\}$ , and  $\{0, 1, 8, 12, 14, 17\}$ .

Computer searches give the largest known value of  $G(p)$ . The current record is  $G(26) = 492$  and the corresponding optimal Golomb ruler has marks

0 1 33 83 104 110 124 163 185 200 203 249 251 258  
314 318 343 356 386 430 440 456 464 475 487 492.

The search took several years and this ruler is not even known to be unique [13, 14]. *Wikipedia* is a good place to look for the latest record values of  $G(p)$ .

Given a Golomb ruler with  $p$  marks, there are  $\binom{p}{2} \sim \frac{1}{2}p^2$  distinct distances one can measure with this ruler. Thus, one expects  $G(p)$  to be roughly at least  $\frac{1}{2}p^2$ . It is a conjecture, with strong empirical evidence, that  $G(p) < p^2$ ; but it is only a conjecture.

Golomb rulers also have a close connection with additive number theory. It is completely outside the scope of this paper to discuss this connection in any depth, and we only state some facts and invite the reader to investigate further.

DEFINITION 4. A subset  $\mathcal{B}$  of integers contained in  $[1, N]$  is called a  $B_2$  basis, if for any two distinct pairs of integers from  $\mathcal{B}$ , say  $a, a'$  and  $b, b'$  we have

$$a + a' \neq b + b'.$$

There is an old conjecture of Erdős and Turán which states that a  $B_2$  basis with  $\lfloor \sqrt{N} \rfloor$  elements can be constructed in  $[1, N]$  for any  $N$ . This is very closely related to the conjecture that  $G(p) < p^2$ . Halberstam and Roth [9] give a comprehensive discussion of additive number theory and the connection with Golomb rulers.

**Perfect rulers on  $\mathbb{N}$**  In this section we study *infinite rulers*. These are sets  $\mathcal{G}$  of nonnegative integers, such any positive integer  $d$  is realized as a distance between some two elements of  $\mathcal{G}$ . If we require, in addition, the representation to be unique, we may speak of infinite perfect Golomb rulers.

DEFINITION 5. A subset  $\mathcal{G}$  of the set  $\mathbb{N}$  of natural numbers is called an *infinite perfect Golomb ruler* if

- 1) for every positive integer  $d$ , there are elements  $a, a' \in \mathcal{G}$  so that  $d = a - a'$ , and
- 2) for every such  $d$  this representation is unique.

It is not entirely clear that such things exist, but in fact they do and also they can be made arbitrarily *thin* (sparse) depending on the choice of a function  $\varphi$ .

THEOREM 2. Let  $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be strictly increasing with  $\varphi(x) \rightarrow \infty$  as  $x \rightarrow \infty$ . There is an infinite perfect Golomb ruler  $\mathcal{G} \in \mathbb{N}$  such that for  $x > x_0 = x_0(\mathcal{G}, \varphi)$

$$\#\{k \mid k \in \mathcal{G}, k \leq x\} \leq \varphi(x). \tag{1}$$

*Proof.* We preview the steps: First we choose a rapidly increasing sequence  $\gamma_k$ ,  $k = 1, 2, \dots$ , and then construct  $\mathcal{G}$  by successively adding the points  $\{\gamma_k, \gamma_k + k\}$ . If a duplication would occur as a result of this addition, then we do not add the pair. Various things have to be proved, for example, that skipping a pair does not result in some integer  $d$  not being realized as the difference of two element of  $\mathcal{G}$ , etc. The details follow.

Choose a strictly increasing function  $\psi(x)$  such that

$$x < \frac{1}{2}\varphi(\psi(x)) \tag{2}$$

and define a sequence  $\{\gamma_k\}$  by

$$\begin{aligned} \gamma_1 &= 0 \\ \gamma_{k+1} &> \psi(k+1) + 2(\gamma_k + k) + 1, \text{ for } k \geq 1. \end{aligned} \tag{3}$$

Define  $\mathcal{A}_1 = \{\gamma_1, \gamma_1 + 1\}$  which of course equals  $\{0, 1\}$ , set

$$\mathcal{A}_{k+1} = \begin{cases} \mathcal{A}_k \cup \{\gamma_{k+1}, \gamma_{k+1} + (k+1)\} \\ \text{if this set } \mathcal{A}_k \cup \{\gamma_{k+1}, \gamma_{k+1} + (k+1)\} \text{ has no duplicate distances} \\ \mathcal{A}_k \text{ otherwise,} \end{cases}$$

and let

$$\mathcal{G} = \bigcup_{k=1}^{\infty} \mathcal{A}_k.$$

First we show that the set  $\mathcal{G}$  satisfies the density condition (1) in the statement of the Theorem. Let  $x > 1$  be given and let  $k_0$  be the largest integer such that  $\gamma_{k_0} \leq x$ . Then

$$\#\{k \mid k \in \mathcal{G}, k \leq x\} \leq 2k_0$$

because the elements of  $\mathcal{G}$  come in pairs:  $\gamma_p$  and  $\gamma_p + p$ . Now

$$k_0 < \frac{1}{2}\varphi(\psi(k_0)) < \frac{1}{2}\varphi(\gamma_{k_0}) \leq \frac{1}{2}\varphi(x).$$

The first inequality follows from (2) and the second and third follow from (3) and the fact that  $\varphi$  is monotonically increasing. Thus the density claim (1) of the Theorem is true.

By construction, there is no duplication of distances in  $\mathcal{G}$ . After all, there is no duplication of distances in any of the sets  $\mathcal{A}_k$ .

It remains to show that every distance  $d$  is realized as a difference of two elements of  $\mathcal{G}$ . It suffices to analyze the pairs not included by our process: When a duplication occurs by inclusion of  $\{\gamma_p, \gamma_p + p\}$ , then we claim that

$$p = a - a' \text{ where } a, a' \in \mathcal{A}_{p-1} \tag{4}$$

that is,  $p$  is already realized as a distance in the set  $\mathcal{A}_{p-1}$ . This would occur, for example, in the following situation: Let  $a$  and  $a'$  be two points in a set  $\mathcal{A}_q$ , for some  $q$  such that  $a - a' > q$ . Then, further along in the process, adding the pair  $\{\gamma_{a-a'}, \gamma_{a-a'} + (a - a')\}$  would surely create a duplication. The claim is that this is essentially the only way it could happen. Now, if this claim is true, then either every distance  $d$  occurs in  $\mathcal{G}$  through the addition of the pair  $\{\gamma_d, \gamma_d + d\}$  or  $d$  occurs already as a distance in the set  $\mathcal{A}_{d-1}$ .

We now prove the assertion (4). Suppose that the addition of the pair  $\{\gamma_p, \gamma_p + p\}$  to the set  $\mathcal{A}_{p-1}$  results in duplications. Because there are no duplications in the set  $\mathcal{A}_{p-1}$  these duplications must involve the points from the pair under discussion. It follows, because of (3), that both points of the pair are larger than any of the points in  $\mathcal{A}_{p-1}$ , and so the possibilities are:

- (i)  $(\gamma_p + p) - \gamma_p = a - a'$
- (ii)  $(\gamma_p + p) - a = \gamma_p - a'$
- (iii)  $(\gamma_p + p) - a = a' - a''$
- (iv)  $\gamma_p - a = a' - a''$

where the numbers  $a, a', a''$  are elements of the set  $\mathcal{A}_{p-1}$ . In cases (i) and (ii) then  $p$  is a difference of some elements in  $\mathcal{A}_{p-1}$ , hence (4) holds. The possibilities (iii) and (iv) cannot occur because the largest element of  $\mathcal{A}_{p-1}$  is at most  $\gamma_{p-1} + (p - 1)$  and from (3) then  $\gamma_p > 2(\gamma_{p-1} + p - 1)$ . But, if either (iii) or (iv) were true, then either  $\gamma_p$  or  $\gamma_p + p$  would be at most twice the largest element of  $\mathcal{A}_{p-1}$ . Thus our claim (4) is shown and the theorem is proved. ■

Thus, thin infinite perfect Golomb rulers do exist. The construction in Theorem 2 does not give a formula for the  $n$ th mark—it just constructs these marks one by one.

It would be interesting to know how *thick* an infinite perfect Golomb can be. In particular is it possible to have

$$\delta_{\mathcal{G}}(x) = \#\{k \mid k \in \mathcal{G}, k \leq x\} \sim \sqrt{x}?$$
 (5)

By arguments similar to the discussion of finite perfect Golomb rulers, it is easy to see that  $\delta_{\mathcal{G}}(x) < \sqrt{2x}$ , and (5) is motivated by the Erdős-Turan conjecture about  $B_2$  bases (it does not follow from nor does it imply the conjecture).

**Minimal spanning rulers** Next we return to rulers of finite length and discuss those that can be used to measure every distance. They differ from Golomb rulers in that there might be a distance that can be measured in two different ways, but we require that every eligible distance can be measured. We call such rulers *spanning*.

**DEFINITION 6.** Let  $\mathcal{S} = \{0 = a_1 < a_2 < \dots < a_p = N\}$  be a set of integers. We say that  $\mathcal{S}$  is a *spanning ruler* on  $[0, N]$  if every integer  $1 \leq d \leq N$  can be expressed as  $d = a - a'$ , with  $a$  and  $a' \in \mathcal{S}$ .

We say that a spanning ruler  $\mathcal{M}$  is *minimal* on  $[0, N]$ , if whenever  $\mathcal{M}'$  is a proper subset of  $\mathcal{M}$  then the set  $\mathcal{M}'$  is not a spanning ruler on  $[0, N]$ .

Minimal spanning rulers obviously exist. Just start with  $\{0, 1, \dots, N\}$  and remove one point at a time until you can't do it anymore.

However, minimal rulers cannot be very *thin*. If  $\mathcal{M}$  is a minimal ruler of length  $N$  and  $p = \#\mathcal{M}$ , then  $\binom{p}{2} = \frac{1}{2}p(p-1) \geq N$ ; so  $p$  is roughly at least  $\sqrt{2N}$ . We now show that we can come fairly close to this lower bound.

**THEOREM 3.** For every integer  $N \geq 4$  there is a minimal spanning ruler  $\mathcal{M}_N \subset [0, N]$  such that

$$2\sqrt{N} - 1 \leq \#\mathcal{M}_N < 2\sqrt{N}$$
 (6)

and the equality on the left side holds only when  $N$  is a perfect square.

*Proof.* The basic idea of the proof can best be seen by an example of a thin minimal ruler for  $N = 100$ . The ruler  $\mathcal{M}_{100}$  is in this case taken to be

$$\mathcal{M}_{100} = \{0, 1, 2, 3, \dots, 9, 20, 30, 40, \dots, 90, 100\}.$$

Notice that the number 10 is *not* included. The number of elements in  $\mathcal{M}_{100}$  is 19 which is equal to  $2\sqrt{100} - 1$ . Every distance  $1 \leq d \leq 100$  is realizable: Represent  $d = 10$  as  $30 - 20$  and other multiples of 10 are just  $d = d - 0$ . If  $d = q \cdot 10 + j$ ,  $1 \leq q, j \leq 9$ , then  $d = (q + 1) \cdot 10 - (10 - j)$ .

Finally, if  $1 \leq d \leq 9$  then  $d = d - 0$ . None of the numbers can be removed. For example,  $d = 7$  cannot be removed because then  $13 = 20 - 7$  would not be realizable. The number 30 cannot be removed because then  $21 = 30 - 9$  would not be realizable. If 10 is included the ruler is not minimal. The actual proof is based on this example although some care must be taken when  $N$  is not a perfect square. Here are the details.

By inspection, when  $N \in \{5, 6, 7, 8\}$  then the minimal spanning rulers satisfying (6) are, respectively:

$$\{0, 1, 3, 5\}, \{0, 1, 4, 6\}, \{0, 1, 4, 5, 7\}, \{0, 1, 4, 6, 8\}.$$

Incidentally, there are no minimal spanning rulers satisfying the condition (6) for  $N \in \{1, 2, 4\}$  and there is one for  $N = 3$ , namely  $\{0, 1, 3\}$ .

Let  $\xi = \lfloor \sqrt{N} \rfloor$  so that  $\xi^2 \leq N < (\xi + 1)^2 = (\xi + 2)\xi + 1$ . We assume that  $\xi \geq 3$ . There are two possibilities:

- ( $\alpha$ )  $N = 0 \pmod{\xi}$  so that  $N = K\xi$ ,  $K = \xi, \xi + 1$ , or  $\xi + 2$ ; (7)
- ( $\beta$ )  $N \neq 0 \pmod{\xi}$  so that  $N = K\xi + \eta$ ,  $K = \xi$  or  $\xi + 1$ , and  $1 \leq \eta < \xi$ .

In case ( $\alpha$ ) take

$$\mathcal{M}_N = \{0, 1, \dots, \xi - 1, 2\xi, 3\xi, \dots, K\xi\} \quad (\xi \text{ is not included})$$

with  $K$  as in (7).

Every distance  $1 \leq d \leq N$  is realized as the following analysis shows: If  $1 \leq d \leq \xi - 1$  then  $d = d - 0$ ; When  $d = \xi$ , then  $d = 3\xi - 2\xi$  since  $2\xi, 3\xi \in \mathcal{M}_N$  for  $\xi \geq 3$ ; For  $d = q\xi$ ,  $q > 1$  then  $d = q\xi - 0$ ; Finally if  $d = q\xi + \eta$ ,  $1 \leq q < K$ ,  $1 \leq \eta < \xi$ , then  $d = (q + 1)\xi - (\xi - \eta)$ .

Also, we see that none of the marks can be removed: The endpoints 0 and  $N$  cannot be deleted because  $N = N - 0$ . The points  $1 \leq d < \xi$  cannot be deleted because of the distance  $\xi + (\xi - d) = 2\xi - d$ . Finally, the points  $q\xi$ ,  $2 \leq q \leq K$ , cannot be deleted because of the distance  $(q - 1)\xi + 1 = q\xi - (\xi - 1)$ .

Counting the marks gives  $\#\mathcal{M}_N = \xi + K - 1$ . Thus, to show (6) we must prove that for  $t = 0, 1, 2$  then

$$2\sqrt{\xi(\xi + t)} - 1 \leq 2\xi + t - 1 < 2\sqrt{\xi(\xi + t)}$$

with equality holding on the left side only when  $t = 0$ . This is done in a straightforward manner by squaring each side of the inequality to eliminate radical expressions.

In case ( $\beta$ ) take

$$\mathcal{M}_N = \{0, 1, \dots, \xi - 1, 2\xi, 3\xi, \dots, K\xi, K\xi + \eta\} \quad (\xi \text{ not included})$$

where  $K, \eta$  are as specified in (7). Again, all the distances  $1 \leq d \leq N = K\xi + \eta$  can be realized: If  $1 \leq d \leq K\xi$  then the argument is the same as in case ( $\alpha$ ); If  $d = K\xi + \delta$ ,  $1 \leq \delta \leq \eta$ ,  $d = (K\xi + \eta) - (\eta - \delta)$ .

None of the marks can be removed: The endpoints cannot be removed because of the distance  $N = N - 0$ . The points  $\xi - c$ ,  $1 \leq c < \xi$ ,  $c \neq \eta$  cannot be removed because of the distance  $\xi + c = 2\xi - (\xi - c)$ .

The point  $\xi - \eta$  cannot be removed because of the distance

$$(K - 1)\xi + \eta = K\xi - (\xi - \eta). \tag{8}$$

However also  $(K - 1)\xi + \eta = (K\xi + \eta) - \xi$  and we see that (8) is the only way to realize the distance  $(K - 1)\xi + \eta$  since  $\xi \notin \mathcal{M}_N$ .

The points  $2\xi, 3\xi, \dots, K\xi$  cannot be removed for the following reason: Let  $\tau$  be such that  $\tau \neq \eta$ ,  $1 \leq \tau < \xi$ . If  $k\xi$  is removed then the distance  $(k - 1)\xi + \tau = k\xi - (\xi - \tau)$  is not realizable. (It can't be realized using the mark  $K\xi + \eta$ .)

Finally,  $\#\mathcal{M}_N = \xi + K$  and to show (6) we must prove the inequality

$$2\sqrt{\xi(\xi + t) + \eta} - 1 < 2\xi + t < 2\sqrt{\xi(\xi + t) + \eta}$$

for  $t = 0, 1$  and  $1 \leq \eta < \xi$ . Again, squaring both sides of each inequality to eliminate radicals and some algebra does the job. ■

Minimal spanning rulers can also be quite *thickly* marked.

THEOREM 4. For any  $N > 0$  there is a minimal spanning ruler  $\mathcal{M}_N$  with

$$\#\mathcal{M}_N > \frac{1}{2}N.$$

*Proof.* A moment's reflection shows that if  $N = 2n$  or  $N = 2n + 1$  then  $\mathcal{M}_N = \{0, 1, \dots, n, N\}$  works. ■

**Acknowledgment** We thank the referees for many valuable suggestions, including a very nice reformulation of the statement of Theorem 3.

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AuQ: Incomplete reference. Book or journal?

**Summary** What is the smallest number of inch marks on a ruler that allow us measure all integral distances? This question motivates our survey of Golomb rulers, perfect rulers, and minimal rulers—different types of rulers for allowing the most measurements with the smallest number of marks.