UNIFORM GROWTH OF POLYCYCLIC GROUPS

ROGER C. ALPERIN

1. Introduction

The Milnor-Wolf Theorem characterizes the finitely generated solvable groups which have exponential growth; a finitely generated solvable group has exponential growth iff it is not virtually nilpotent. Wolf showed that a finitely generated nilpotent by finite group has polynomial growth; then extended this by proving that polycyclic groups which are not virtually nilpotent have expontial growth, [8].

On the other hand, Milnor, [5], showed that finitely generated solvable groups which are not polycyclic have exponential growth. In both approaches exponential growth can be deduced from the existence of a free semigroup, [1, 6].

In this article we elaborate on these results by proving that the growth rate of a polycyclic group Γ of exponential growth is uniformly exponential. This means that base of the rate of exponential growth $\beta(S,\Gamma)$ is bounded away from 1, independent of the set of generators, S; that is, there is a constant $\beta(\Gamma)$ so that $\beta(S,\Gamma) \geq \beta(\Gamma) > 1$ for any finite generating set. The growth rate is also related to the spectral radius $\mu(S,G)$ of the random walk on the Cayley graph, with the given set of generators, [3].

The exponential polycyclic groups are an important class of groups for resolving the question of whether or not exponential growth is the same as uniform exponential growth, since they are in a sense very close to groups of polynomial growth. The ideas used here for polycylic groups take advantage of their linear and arithmetic properties. These may be important tools for proving uniform growth for other classes of groups. Other methods for proving uniform growth take advantage of special properties of presentations; for example, an excess of the number of generators over relations by at least 2 ensures the existence of a subgroup of finite index which maps onto a non-abelian free group. Here, of course we can not map to free groups, but we are in a sense able to map to a non-abelian free semigroup.

2. Exponential Growth

The growth of a group is measured in the following way. Choose a finite generating set, S, for the group Γ; define the length of an element as $\lambda_S(g) = min\{\ n \mid g = s_1 \cdots s_n, s_i \in S \cup S^{-1}\}$. The growth function $\beta_n(\Gamma, S) = |\{g \mid \lambda_S(g) \leq n\}|$ depends on the chosen generating set, but as Milnor showed, whether or not it is exponential is independent of the generating set. A group has exponential growth if the growth rate, $\beta(S,\Gamma) = limit_{n\to\infty}\beta_n(S,\Gamma)^{\frac{1}{n}}$ is strictly greater than 1, for some (any) generating set. The growth rate is the reciprocal of the radius of convergence of the growth series, the generating function with coefficients given by $\beta_n(S,\Gamma)$.

If we have another finite generating set $T = \{t_1, \cdots, t_m\}$ for Γ, and $max_j\lambda_S(t_j) \leq L$, $max_i\lambda_T(s_i) \leq L$, then $\beta_n(S,\Gamma) \leq \beta_{Ln}(T,\Gamma)$ and also the symmetric inequality.

It then follows that $\beta(S,\Gamma)^L \leq \beta(T,\Gamma)$ and $\beta(T,\Gamma)^L \leq \beta(S,\Gamma)$ so that exponential rate for one set of generators means exponential rate for any system of generators.

Suppose that Γ has a free semigroup on two generators, say a, b. Let $T = S \cup \{a, b\}$; then, we have the interesting feature, using estimates as above, that $\beta(S,\Gamma)^L \geq \beta(T,\Gamma) \geq 2$, where L is the maximum of the length of the elements a, b in terms of the generators S.

For a group with exponential growth we consider

$$\beta(\Gamma) = inf_S \beta(S,\Gamma).$$

If $\beta(\Gamma) > 1$ then Γ is said to have uniform (exponential) growth.

Proposition 2.1. *Given a finitely generated group Γ. Suppose there is a constant L so that for every finite generating set S there are two elemnts a_S, b_S which generate a free semigroup and whose lengths in that generating set are bounded by L, then Γ has uniform exponential growth.*

Proof: It follows immediately from above, $\beta(S,\Gamma) \geq 2^{\frac{1}{L}}$.

The following uses the fact that given a generating set S for Γ the set of elements of the subgroup $\mathcal{K}$, a subgroup of index d, which are words in S of length at most $2d-1$ give a generating set for $\mathcal{K}$.

Proposition 2.2 ([7])**.** *If a group Γ has a subgroup $\mathcal{K}$ of finite index d, then $\beta(\Gamma) \geq \beta(\mathcal{K})^{\frac{1}{2d-1}}$.*

The following is immediate from the definition and Proposition 2.2 on passage to finite index.

Corollary 2.3. *If a group Γ has a subgroup of finite index which has a homomorphic image which has uniform growth, then Γ has uniform growth.*

Let A be an automorphism of infinite order of a finitely generated abelian group $\mathcal{V}$. The group $\mathcal{V}_A = \mathcal{V} \rtimes_A \mathbf{Z}$ is the split extension of $\mathcal{V}$ by the automorphism A. In the split extension, Γ_A, let t denote the element which maps to the generator of $\mathbf{Z}$, and which acts via conjugation on $\mathcal{V}$ via the automorphism A. Let $M(v)$ denote the cyclic submodule generated by $v \in \mathcal{V}$. Let $\mathcal{V} \otimes_{\mathbf{Z}} \mathbf{C}$ denote the extension of scalars to the complex numbers, $\mathbf{C}$.

Lemma 2.4. *Let A be an automorphism of a finitely generated abelian group $\mathcal{V}$. If A has an eigenvalue $|\lambda| \geq 2$ on $M(v) \otimes_{\mathbf{Z}} \mathbf{C}$ then the split extension, $\mathcal{V}_A$, contains the free semigroup generated by tv and v.*

Proof: Let $v_i = t^i v t^{-i}$. Consider the set $W = \{w \mid w = \prod_{i=1}^n v_i^{e_i}, e_i \in \{0,1\}, n \geq 1\}$. Let $U = \{t^i \mid i \geq 1\}$. Set $X = WU$. Consider the sets tvX and tX. By a calculation, $t(wt^i) = t(\prod_{i=1}^n v_i^{e_i})t^{-1}t^{i+1} = \prod_{i=1}^n v_{i+1}^{e_i} t^{i+1}$ and also $tv(wt^i) = tvt^{-1}twt^{-1}t^{i+1} = v_1 \prod_{i=1}^n v_{i+1}^{e_i} t^{i+1}$; thus, $tX \cup tvX \subseteq X$

Furthermore, these two subsets tvX and tX do not meet. For if they did then there would be an equality $tw_1t^i = tvw_2t^j$ for some $w_1, w_2 \in W$. From the calculations above and the uniqueness of the power of t in U it follows that $i = j$ and then also that $tw_1t^{-1} = v_1(tw_2t^{-1})$. This gives rise to two polynomials W_1, W_2 in A, the matrix of conjugation by t, such that $AW_1(A)v = Av + AW_2(A)v$. But since v generates the cyclic module this means $AW_1(A) - AW_2(A) = A$ or simply $W_1(A) - W_2(A) = I$ where the coefficients of W_1, W_2 are 0 or 1, without constant term. Consequently this identity is also valid for every eigenvalue λ of A and therefore $W_1(\lambda) - W_2(\lambda) = 1$ so that for some $m \geq 1$ (highest power in W_1 or W_2), $\lambda^m = \epsilon_0 + \epsilon_1\lambda^1 + \cdots + \epsilon_{m-1}\lambda^{m-1}$, $\epsilon_j \in \{0, \pm 1\}$, so that

$$|\lambda|^m \leq 1 + |\lambda| + |\lambda|^2 + \cdots + |\lambda|^{m-1} \leq \frac{|\lambda|^m - 1}{|\lambda| - 1}.$$

This however is impossible for $|\lambda| \geq 2$.

Now from the disjointness of $X_1 = tX$ and $X_2 = tvX$, and also $X_1 \cup X_2 \subset X$, it now easily follows that t, tv generate a free semigroup. Any distinct words u_1, u_2 in t, tv without loss of generality begin on the left with t, tv respectively, so $u_1X \subset X_1, u_2X \subset X_2$ are different.

3. Abelian By Cyclic

Lemma 3.1. *Suppose that a finitely generated group Γ has a normal free abelian subgroup N, of rank r, with cyclic quotient generated by T having an eigenvalue of its characteristic polynomial of absolute value at least 2, then Γ has uniform exponential growth.*

Proof: Given a generating set S, Consider N as a $Z[T]$ module with T acting by conjugation.

Consider the collection S_0 of generators of S, which are in N, together with all commutators of generators. Some generator, or its inverse, say t, maps to a non-zero power of T and so it also has an eigenvalue of absolute value at least two. The action of $t \in S$ on N satisfies its characteristic polynomial of degree r, so in fact the finite set of conjugates of elements of S_0 by powers of t^q for $0 \leq q < r$, yields S_1 a generating set for the t invariant subgroup N_1.

We prove the proposition by induction on $k = rank_{\mathbf{Z}}(N) - rank_{\mathbf{Z}}(N_1)$. If $k = 0$ then N_1 is of finite index in N. The eigenvalue of absolute value at least 2 will be supported on some cyclic submodule generated by one of the generators, since the least common multiple of all the annihilators of the generators is the annihilator of the module, N_1, which is the characteristic polynomial of T. Thus, by Lemma 2.4, one of these generators $v \in S_1$ of N_1 generates a cyclic submodule, so that t has eigenvalue at least 2 on $Z[T]v \otimes_{\mathbf{Z}} \mathbf{C}$. The free semigroup generators t, tv have lengths bounded by $L = 3 + 2r$, so $\beta(\Gamma) \geq 2^{\frac{1}{L}}$.

Suppose that $k > 0$, then N/N_1 has an action of t, as well as N_1; both groups are non-zero and have smaller rank than N. To see that, suppose that $N_1 = \{0\}$ then all commutators of generators are trivial and then Γ is abelian, which is of polynomial growth; this is impossible by Lemma 2.4, since Γ must have exponential growth, and so is not of polynomial growth. Thus N_1 is non-zero. Furthermore, the torsion subgroup of N/N_1 is invariant under t so that we may assume it is free abelian. On one or the other of these N_1 or N/N_1, the action of t has an eigenvalue at least 2. To see this we choose rational bases for N_1 and its complement, obtained by lifting a basis for N/N_1. The characteristic polynomial of t on N_1 is a factor of that on N, with the complementary factor being given by the action on the quotient group N/N_1.

If the eigenvalue at least 2 is supported on N_1 then it must occur in the exponent for this submodule, and hence is supported on one of the generators; so we obtain the free semigroup as above with semigroup generators of length bounded by a function of the rank and hence we have free semigroup generators for Γ with generators of bounded length.

Otherwise, consider the subgroup Γ_1 which is the inverse image of the subgroup generated by t in $\mathbf{Z}$; it contains N_1 as a normal subgroup. $\overline{\Gamma}_1 = \Gamma_1/N_1$ has a (non-abelian) free semigroup by Lemma 2.4, so by induction, $\overline{\Gamma}_1$ is of uniform exponential growth and hence also Γ by Corollary 2.3, since $\overline{\Gamma}_1$ is a homomorphic image of a subgroup of finite index.

Proposition 3.2 (Kronecker)**.** *An algebraic integer with all of its algebraic conjugates having absolute value 1 is a root of unity.*

Proof: It generates a compact yet discrete group in the standard diagonal embedding of the number field (see below).

Let $\mathcal{O}$ be the ring of integers in a number field F.

Proposition 3.3. *Given $k > 1$, there is a constant $\mu \geq 1$ depending on $\mathcal{O}$ so that for any non-zero element $x \in \mathcal{O}$, which is not a root of unity, there is an embedding into the complex numbers $\rho : F \to \mathbf{C}$ so that $|\rho(x)^m| \geq k$ for all $m \geq \mu$.*

Proof: Consider the collection of all elements of $\mathcal{O}$ so that in every embedding ρ into the complex numbers $\rho(x) \leq k$. This is a compact set for the diagonal embedding $\Delta : F \to \mathbf{R}^r \times \mathbf{C}^s$, across the real and imaginary places. However, the image of $\mathcal{O}$ is discrete. Consequently, the collection of all elements $x \in \mathcal{O}$ such that $1 \leq |\rho(x)| \leq k$, for all embeddings is a finite set, T_k. Let $\kappa = min_{x \in T_k} max_\rho |\rho(x)|$; choose μ large enough so that so that $\kappa^\mu > k$.

Given $x \in \mathcal{O}$, non-zero and not a root of unity, choose an embedding ρ, by Kronecker's Lemma, so that $|\rho(x)| > 1$, but so that ρ achieves the maximal absolute value among all embeddings. This is possible since the absolute value of the norm, $|N(x)| = \prod_\rho |\rho(x)|$, is an integer greater than 1. If $|\rho(x)| \geq k$ we're done, so we may assume $1 < |\rho(x)| < k$ for all embeddings ρ. Thus, $\kappa \leq |\rho(x)| < k$, and hence the value μ suffices.

Suppose that a finitely generated group Γ has a normal free abelian subgroup $\mathcal{V}$, of rank r, with nilpotent quotient $\mathcal{N}$. On a subgroup of finite index, the action of $\mathcal{N}$ on $\mathcal{V} \otimes_{\mathbf{Z}} \mathbf{C} = \mathbf{C}^r$ can be put in upper-triangular form, $\rho(\mathcal{N})$, by Malcev's Theorem. The elements $M = \{\rho(g) \mid g \in \mathcal{N},\ \text{all eigenvalues are of absolute value } 1\}$ forms a subgroup of $\rho(\mathcal{N})$. The inverse image under ρ gives a subgroup $\mathcal{M}$ of $\mathcal{N}$.

Proposition 3.4 (*cf.* [6])**.** *Suppose that a finitely generated group Γ has a normal free abelian subgroup $\mathcal{V}$, of rank r, with nilpotent quotient, $\mathcal{M}$, where every element acts so that all its eigenvalues are of absolute value 1, then Γ is nilpotent by finite and hence of polynomial growth.*

Proof: With the notation from above we construct the homomorphism $\Delta : \rho(\mathcal{M}) \to (S^1)^r$ mapping each matrix to its diagonal part. Since each of the diagonal entries satisfies an integral polynomial, and all of its conjugates are of absolute value 1 then by Proposition 3.3 the diagonal entries are roots of unity. Since the image of Δ is finitely generated, it is of finite order. Thus $\rho(M)$ has a subgroup $\rho(U)$ of finite index which is a group of unipotent matrices. Hence, $U/\mathcal{V}$ which is a subgroup of finite index of $\mathcal{M} = \Gamma/\mathcal{V}$ is also nilpotent. Moreover, we claim that U is nilpotent and therefore it follows that $\mathcal{M}$ is nilpotent by finite.

To see that U is nilpotent we continue a central series from $\mathcal{V}$; denote the homomorphism $\phi : U \to U/\mathcal{V}$. Take a central series $U_i = \phi^{-1}(Z_i)$ corresponding to Z_i of $U/\mathcal{V}$, so that $U_r = U$.

Next, let $\{v_1, \cdots, v_r\}$ be the basis that makes $\rho(U)$ unipotent. Hence $\mathcal{V}$ is contained in the **C**-span of $v_1, \cdots, v_r$. Let $V_k = \mathcal{V} \cap span_{\mathbf{C}}\{v_1, \cdots, v_k\}$. Then $V_r = \mathcal{V}$ and this gives a series $V_i \subset V_{i+1}$ extending the series of U_i. Moreover, this is a central series, because V_{i+1}/V_i belongs to the center of U/V_i. To see this, observe that $\rho(u)V_{i+1} \subset V_{i+1}$. Also, $\rho(u)v_{i+1} = v_{i+1} \ mod \ span_{\mathbf{C}}\{v_1, \cdots, v_i\}$ for any element $u \in U$. Thus, for $v \in V_{i+1}$, then $\rho(u)v - v \in V_i$.

Corollary 3.5. *Suppose that a finitely generated group Γ has a normal free abelian subgroup $\mathcal{V}$, of rank r, with nilpotent quotient, $\mathcal{N}$. Then Γ is nilpotent by finite or some element t of $\mathcal{N}$ acts on $\mathcal{V}$ so that not all its eigenvalues are of absolute value 1. In this later case Γ is of exponential growth.*

Proof: In the first case the result follows immediately from Proposition 3.4. In the later case, the result follows from Lemma 2.4, using a large enough power of t or its inverse to get an eigenvalue of absolute value at least 2.

Theorem 3.6 (Abelian by Cyclic Alternative)**.** *Suppose that a finitely generated group Γ has a normal free abelian subgroup $\mathcal{V}$, of rank r, with cyclic quotient. Then either Γ is of polynomial growth or is of unform exponential growth.*

Proof: If all the eigenvalues for the action are roots of unity then Γ is nilpotent by finite by Proposition 3.4. So we shall assume otherwise.

Given a generating set S for Γ we choose a generator t in S mapping non-trivially to say T^m, when the infinite cyclic quotient is generated by T. The action of T on the subgroup $\mathcal{V}$ satisfies its characteristic polynomial which is of degree r. We can assume that the matrix of T^{nm} has an eigenvalue of absolute value at least 2, for some n (we may have to replace t by t^{-1} to accomplish this) where n depends not on the generating set but rather on the ring of integers in the number field generated by the roots of the characteristic polynomial of T, using Proposition 3.3.

Next, consider the subgroup of Γ of finite index $\Gamma_1 =< \mathcal{V}, t^n >$. It now follows by Proposition 2.2 and Lemma 3.1 that Γ has uniform exponential growth.

4. Polycyclic

It is well-known that finitely generated polycyclic Γ, [4], group has an embedding into $GL_m(Z)$. Whenever we have such an embedding and a normal subgroup $\mathcal{N}$ we consider the action of the quotient as a group of automorphisms of the **Z**-linear span of $\mathcal{N}$ in $M_m(\mathbf{Z})$. Consequently each automorphism has eigenvalues which belong to a ring of integers in a number field of degree at most m^2 over the rationals.

Suppose that Γ is a polycyclic group, which is an extension of a nilpotent normal subgroup $\mathcal{N}$ by a free abelian group $\mathcal{A}$ of finite rank. Choose a splitting of $\mathcal{A}$ into Γ and consider the action of $\mathcal{A}$ on Γ by inner automorphisms, using the matrix representation as above. This action gives a sequence of actions on the terms of the lower central series of $\mathcal{N}$, in which each element satisfies a polynomial of degree at most m^2, on each term of the series. We refer to this as the action on $\mathcal{A}$ on $\mathcal{N}$. Furthermore, for the action of Γ and $\mathcal{A}$ on the lower central series of $\mathcal{N}$, there is an eigenvalue not of absolute value 1, or else the group Γ is nilpotent by finite, [8], [2], *appendix*.

Lemma 4.1. *Suppose that for the action $\mathcal{A}$ on $\mathcal{N}$, every non-identity element, t of $\mathcal{A}$, either t or t^{-1}, has some eigenvalue of absolute value at least 2, then Γ has uniform exponential growth.*

Proof: We use induction on the Hirsh rank of $\mathcal{N}$, $h = h(\mathcal{N})$, the number of infinite-cyclic factors in a cyclic series for $\mathcal{N}$. Consider a set of generators S for Γ. Let $\mathcal{N}_1$ be the normal subgroup of $\mathcal{N}$, normally generated by the set S_0 of all the generators of S belonging to $\mathcal{N}$, together with all commutators of generators from S. Since each element from Γ satisfies a polynomial of degree m^2, the subgroup $\mathcal{N}_1$ is then generated by S_0, and the conjugates of this normal set of generators by $t^q, q < m^2$, using all the generators $t \in S$. Denote this set of generators of $\mathcal{N}_1$ by S_1.

It follows from the hypothesis, that Γ has exponential growth, and hence it could not be virtually abelian; thus $\mathcal{N}_1$ is non-trivial.

Suppose this nilpotent normal subgroup $\mathcal{N}_1$ has the same rank as $\mathcal{N}$ then it will provide a free semigroup using the terms of the lower central series and the generating set T. Thus, the extension of $\mathcal{N}_1$ by $\mathcal{A}$ is of finite index in Γ. As we show next, this is sufficient to finish the proof.

Let $\mathcal{H} = \mathcal{N}_1$. Consider the quotients of the lower central series $\mathcal{H}_k/\mathcal{H}_{k+1}$ of $\mathcal{H}$. These are finitely generated abelian groups and generated by the left normed commutators of weight k in the generating set S_1 for $\mathcal{H}$. Consequently, some generator from S acts on the lower central series quotients so as to have an eigenvalue which is not a root of unity. Since we have bounds on the lengths of a generating set of the lower central series in terms of its weight and the lengths of elements of S_1 elements in terms of S we have a free semigroup, where the lengths of the generators are bounded in terms of m and $h(\mathcal{N})$.

We continue with the induction argument.

If the rank of $\mathcal{N}_1$ is strictly smaller than the rank of $\mathcal{N}$, and $\mathcal{A}$ is acting so that some element has an eigenvalue at least 2 in absolute value then we finish the proof as in the previous case.

Otherwise, $\mathcal{A}$ acts so that every element has all eigenvalues which are roots of unity. In this case the extension of $\mathcal{N}_1$ by $\mathcal{A}$ is virtually nilpotent. Now if both $\mathcal{N}/\mathcal{N}_1$ by $\mathcal{A}$ and $\mathcal{N}_1$ by $\mathcal{A}$ are nilpotent by finite then we can construct the representation of $\mathcal{A}$ on $\mathcal{N}$ which is the sum of these two actions and hence has only eigenvalues which are roots of unity. In this case Γ is nilpotent by finite, by Proposition 4.2. This is impossible since under the hypotheses on Γ it has exponential growth. Hence, $\mathcal{N}/\mathcal{N}_1$ admits an action of $\mathcal{A}$; the action here is such that every generator has an eigenvalue of absolute value at least 2, and thus, by induction, the group extension of $\mathcal{N}/\mathcal{N}_1$ by $\mathcal{A}$ is of uniformly exponential growth; this group is a quotient of Γ; hence, also Γ has exponential growth.

Proposition 4.2 (*cf.* [8])**.** *Suppose that Γ is an extension of a nilpotent group $\mathcal{A}$ by a nilpotent group $\mathcal{N}$ for which the action of $\mathcal{A}$ on the lower central series $\mathcal{N}$, has every element with all eigenvalue of absolute value equal to one, then Γ is nilpotent by finite..*

Proof: Use induction on the nilpotency of $\mathcal{N}$ and Proposition 3.4.

Theorem 4.3 (Polycyclic Alternative). *Suppose that Γ is a polycyclic by finite group, then either Γ has polynomial growth or has uniform exponential growth.*

Proof: By using the Lie-Kolchin-Mal'cev Theorem, [4], and passing to a subgroup of finite index, Γ can be realized as a torsion free group of upper triangular matrices, which has commutator subgroup, $\mathcal{N}$, which is nilpotent, and represented by unipotent matrices. We may assume that Γ is also poly-$\mathbf{Z}$ by passing again to a subgroup of finite index. The subgroup $\mathcal{N}$ is finitely generated, since polycyclic groups satisfy the maximum condition on subgroups (Noetherian).

We consider the inverse image, $\mathcal{N}_1$, in Γ, of the set of elements of Γ where the elements act with all eigenvalues, roots of unity; this subgroup, $\mathcal{N}_1$, is nilpotent by finite by Proposition 3.4; it contains the commutator subgroup of Γ; it is normal in Γ with quotient which is the finitely generated abelian group, $\mathcal{A}$.

The inverse image Γ_1 in Γ of the maximal torsion-free summand $\mathcal{B}$ in $\mathcal{A}$ is of finite index in Γ. If the torsion free rank of $\mathcal{B}$ is zero, then of course Γ is nilpotent by finite. If the torsion free rank of $\mathcal{B}$ is nonzero, It suffices by Proposition 2.2 to show Γ_1 is of uniform exponential growth.

We then may assume then that Γ_1 is nilpotent by free abelian, where every non-identity element of the non-trivial free abelian group $\mathcal{A}$ acts with an eigenvalue which is not a root of unity. Choose n, by Proposition 3.3 so that the eigenvalues of every non-trivial element of $\mathcal{A}$ or its inverse has n^{th} power which has an eigenvalue which has absolute value greater than or equal to 2. The n^{th} power of all elements in $\mathcal{A}$ forms a subgroup of finite index in $\mathcal{A}$ and its inverse image is a subgroup of finite index in Γ_1 satisfying the hypotheses of Lemma 4.1. Hence it is of uniform exponential growth.

5. Solvable

We denote the terms of the derived series of Γ by $\Gamma^{(k)}$. When $\Gamma^{(k-1)}/\Gamma^{(k)}$ are finitely generated for all $k \leq m$ then $\Gamma/\Gamma^{(m)}$ is polycyclic. The following is an immediate corollary from Theorem 4.3.

Corollary 5.1. *Suppose that Γ is a finitely generated group, for which some quotient, $\Gamma/\Gamma^{(m)}$, is polycyclic but not nilpotent by finite then Γ has uniform exponential growth.*

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E-mail: alperin@mathcs.sjsu.edu, Department of Mathematics and Computer Science, San Jose State University, San Jose, CA 95192 USA