RELATIONSHIPS BETWEEN SPECTRAL PEAK FREQUENCIES OF A CAUSAL AR(P) PROCESS AND ARGUMENTS OF ROOTS OF THE ASSOCIATED AR POLYNOMIAL

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ABSTRACT

RELATIONSHIPS BETWEEN SPECTRAL PEAK FREQUENCIES OF A CAUSAL AR(P) PROCESS AND ARGUMENTS OF ROOTS OF THE ASSOCIATED AR POLYNOMIAL

by Abera R. Jiru

This project attempts to assess the relationships between the arguments of the roots of an AR polynomial and the spectral peak frequencies of a causal $\mathrm{AR}(p)$ process. First and foremost, properties of causal $\mathrm{AR}(2)$and $\mathrm{AR}(3)$ processes are thoroughly examined. In addition, the behavior of their spectral densities and the frequencies that maximize the spectral densities are examined. Any apparent relations that these frequencies and the arguments of the roots of the associated AR polynomial might have are also investigated. An attempt is also made to generalize any findings to higher order causal autoregressive processes.

The study shows that the argument of the root of an AR polynomial of a causal $\mathrm{AR}(p)$ process either tends to or is equal to the spectral peak frequency when the modulus of the root tends to one. The spectral peak frequencies and the arguments of the roots of the AR polynomial are also found to approximately the same when $|\phi_p| \approx 1$. In addition, when $\phi_j = 0$, for all $j < p$ regardless of the value of ϕ_p, the spectral peak frequencies and the arguments of the roots of AR polynomial are found to be the same.

DEDICATION

I am indebted to Professor Steven Crunk for his valuable advice, comments and suggestions at every stage of the study. My deepest gratitude is also due to my wife Bethelihem Gudeta for her continuous support.

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CHAPTER 1

Introduction

The notion that a time series is stationary is fundamental in time series analysis. In many literatures the term stationary is referred to as weakly stationary, stationary in a wide sense or second order stationary. The stationarity of time series depends on the existence of the mean function and the autocovariance function of the process. That is, the process is stationary when the mean function is a constant (independent of the time index) and the autocovariance function exists and depends only on the time lag (see Section 2.4). On the other hand, as stated in Anderson (1971) the idea of strict stationary is "that the behavior of a set of random variables at one time is probabilistically the same as the behavior of a set at another time." We will discuss these properties thoroughly in later chapters.

In time series analysis, the autoregressive model is one of the most useful time series models. In dealing with an autoregressive model of order p, AR(p), knowing properties of the model is very important. This is because theoretical properties are the reference frame to which sampling properties are compared. Shumway and Stoffer (2006) stated the autoregressive model as a model in which the current value of the series, x_t, is explained by p past values, $x_{t-1}, x_{t-2}, \dots, x_{t-p}$, where p denotes the number of steps into the past required to forecast the current value and $\{x_t\}$ is a stationary process. As opposed to an ordinary regression model, in an autoregressive model the current value of the dependent variable is influenced by its own past values, thus the name autoregressive. This autoregressive model is given by $x_t = w_t + \sum_{j=1}^{p} \phi_j x_{t-j}$, where the ϕ_j are autoregressive model coefficients and w_t is white noise with variance σ_w^2 (see Chapter 2 for details).

One of the indispensable properties that some autoregressive models satisfy is causality. The autoregressive model is said to be causal if it satisfies the condition that the roots of the polynomial $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j$ lie outside of the unit circle. This polynomial is referred to as the AR polynomial of the process. An alternative condition for checking causality, which is completely based on the autoregressive coefficients, is given in Pandit and Wu (1983). We will discuss this condition in detail in later chapters.

There are two approaches in time series analysis: time domain analysis and frequency domain analysis. Wei (2006) stated that the two approaches are theoretically equivalent. Under the assumption of absolute summable autocovariance function of a stationary time series $\{x_t\}$, the spectral density, $f(\cdot)$, of the process is a function that characterizes the process in the frequency domain. The autocovariance function is one of the characteristics of the time series under the time domain. Shumway and Stoffer (2006) stated that "The autocovariance function and the spectral density contain the same information," under the assumption of absolute summable autocovariance function.

One of the uses of spectral analysis is to determine the neighborhood of the frequency at which the values of f are large (Newton and Pagano, 1983). They also concluded that such frequency components are important in explaining the variability in the time series $\{x_t\}$. Wei (2006) also stated that "A peak in the spectrum indicates the important contribution to the variance from the components at frequencies in the corresponding interval." This means that the spectral peak frequencies have a significant role in spectral analysis. By spectral peak frequency we meant the frequency at which the spectral density of the process is maximized. Such a frequency is also called peak frequency (Ensor and Newton, 1988 and Muller and Prewitt, 1992). The peak of the spectral density could be either a local or an absolute maximum point. The area under the spectral density within a given frequency interval is the proportion of variance of the process explained by the frequencies in the interval. The total area under the spectral density over the interval [-π, π] is the variance of the process.

Now suppose $z_j = r_j e^{i\theta_j}$ is one of the roots of the AR polynomial $\phi(z)$ of a causal AR(p) process, where r_j is the modulus and θ_j is the argument of the root. Any root of the AR polynomial of the process is handled under the time domain approach, so do the modulus and argument of the root. There are cases in which the modulus of a root tends one (or equivalently the root is near the unit circle) and the argument of the root either tend to or equal to one of the spectral peak frequencies. Therefore, it reasonable to investigate the conditions under which θ_j is either near or equal to one of the spectral peak frequencies by way of exploring the lower order autoregressive models first and then try to generalize the findings to the upper order autoregressive models. This is the objective of this Writing Project.

Important definitions, notation and one theorem with its proof are presented in Chapter 2. Descriptions of fundamental concepts that are helpful in later chapters are also given in this chapter. Chapter 3 focuses on exploring the relationships between arguments of the roots of the AR polynomial and spectral peak frequencies for autoregressive model of order p for $p \geq 2$. The behaviors of the spectral densities of the autoregressive processes are also discussed in this chapter. We also proved two important theorems. In Chapter 4, some generalizations are drawn based on the results in Chapter 3. Application of results is also given.

CHAPTER 2

Definitions, Notation and Fundamental Concepts

In this chapter we define some basic terms and describe notation and fundamental concepts that we are going to use later. In addition, we state some important assumptions under which the definitions are valid. We will also prove one theorem.

2.1 Time series

Things that change with time are of special interest to many researchers. Measurements, aspects, features or characteristics of things, objects, persons, etc., that fluctuate with time or over a period of time constitute a time series. The average monthly temperature of San Jose City collected for the past hundred years, average annual amount of rainfall of San Jose City collected for the last fifty years, and Dow Jones Industrial Average (DJIA) collected over some period of time are examples of time series. If observations are collected only at the time points ±1, ±2, ±3, …, then the time series is referred to as discrete time series while if the observations are recorded continuously over some time interval, then the series is called continuous time series. We emphasize discrete time series in this study. Henceforth by time series we mean discrete time series in this Writing Project.

The statistical method that deals with the analysis of time series data is called time series analysis. In other statistical analysis, variables are usually assumed to be independent and the order in which observations are collected is not of much importance. Unlike other statistical analysis, in time series analysis observations are dependent on one another and also collected in an orderly manner. The dependence of successive observations is what makes time series data different from other statistical sets of data. Throughout this chapter and successive chapters we think of x_t as either a random variable or an observation depending on the context. We define a time series as follows.

Definition 1 Let x_t be a random variable indexed by a time point t, where t = 0, ±1, ±2, ±3, …. Then a sequence, $\{x_t\}$, is defined as a time series.

2.2 Mean Function and Autocovariance Function

Given a time series $\{x_t\}$, the mean and autocovariance functions are important characteristics of the process and play a significant role in time series analysis. Their definitions and notation are given as follows.

Definition 2 The mean function, μ_{x_t}, of the time series (or process) $\{x_t\}$ is defined as $\mu_{x_t} = E(x_t)$, provided that the expected value exists.

Definition 3 The autocovariance function, $\gamma_x(s,t)$, of the time series (or process) $\{x_t\}$ is defined as a function

$$\gamma_x(s,t) = Cov(x_s, x_t) = E\left((x_s - \mu_{x_s})(x_t - \mu_{x_t})\right), \ \forall s, \forall t, \tag{1}$$

provided that the expectation exists.

Example 1 Consider $x_t = \beta_1 + \beta_2 t + w_t$, where β_1 and β_2 are known constants and w_t are uncorrelated random variables with zero mean and variance σ_w^2. Then the mean and autocovariance functions are calculated as

$$\mu_{x_t} = E(x_t) = E(\beta_1 + \beta_2 t + w_t) = \beta_1 + \beta_2 t$$

$$\gamma_x(s,t) = Cov(x_s, x_t) = E\left((x_s - \mu_{x_s})(x_t - \mu_{x_t})\right)$$

$$= E(x_s x_t) - \mu_{x_s}\mu_{x_t}$$

$$= E((\beta_1 + \beta_2 s + w_s)(\beta_1 + \beta_2 t + w_t)) - (\beta_1 + \beta_2 s)(\beta_1 + \beta_2 t)$$

$$= E(w_s w_t)$$

$$= \begin{cases} \sigma_w^2 & if\ s = t \\ 0 & if\ s \neq t \end{cases}$$

2.3 White Noise Process

Some time series have special characteristics such as their mean function values vanish for all time points t; they have constant variances; and the values of their autocovariance functions vanish at distinct time points. A time series having such characteristics is referred to as a white noise process. In Example 1, $\{w_t\}$ is a white noise process. We give a formal definition of a white noise process as follows.

Definition 4 Suppose that $\{w_t\}$ is a process such that the following conditions hold.

i) $\gamma_w(s,t) = 0, \ \forall s \neq t$;

ii) $\gamma_w(t,t) = \sigma_w^2 < \infty, \forall t$; and (2)

iii) $\mu_{w_t} = 0, \ \forall t$.

Then the process is referred to as a white noise process. The random variable w_t is referred to as white noise.

The definition indicates that a white noise process is a sequence of uncorrelated random variables with zero mean and constant variance. In most applications and simulation problems the white noise process is assumed to be Gaussian.

2.4 Stationary Time Series

Some time series are simple in their structures while others are complex. An important property of a time series is the one in which the behavior of the time series at one time point is probabilistically the same as its behavior at another time point. That is, the joint distribution of every collection $\{x_{t_1}, x_{t_2}, \ldots, x_{t_k}\}$ is equal to the joint distribution of $\{x_{t_1+h}, x_{t_2+h}, \ldots, x_{t_k+h}\}$, $\forall k = 1, 2, \ldots, \forall t_1, t_2, \ldots, t_k$, and $\forall h = 0, \pm 1, \pm 2, \ldots$. This property is referred to as strictly stationary. Due to the fact that this property is stronger than necessary for many applications, a milder version of this property, which is considered as regularity condition and often times assumed in the analysis of time series, is usually used. The following definition gives detailed description of this milder version of the property.

Definition 5 Suppose that $\{x_t\}$ is a time series that satisfies the conditions

i) $\mu_{x_t} = \mu, \ \forall t$;

ii) $\gamma_x(s,t) = \gamma_x(h), \ \forall h = s-t$; and (3)

iii) $\gamma_x(t,t) = Var(x_t) < \infty, \ \forall t$.

Then the process $\{x_t\}$ is said to be weakly stationary. Henceforth, the term stationary will be taken to mean weakly stationary.

Suppose that $\{x_t\}$ is a stationary process. Then the autocovariance function of the process satisfies the following properties (see Brockwell and Davis, 1991 for proofs).

i) $\gamma_x(0) \geq 0$;

ii) $|\gamma_x(h)| \leq \gamma_x(0), \quad \forall h$; and

iii) $\gamma_x(h) = \gamma_x(-h), \quad \forall h$.

A strictly stationary process with finite variance is always stationary. And also if the process is Gaussian, then strictly stationary and weakly stationary are equivalent.

Example 2 The process in Example 1 is not stationary because the mean function of the process, $\mu_{x_t} = \beta_1 + \beta_2 t$, depends on the time index t.

Example 3 Consider the process $x_t = 0.5x_{t-1} + w_t$, where w_t is white noise with variance one. Then iterating the process backwards r times, we get

$$x_t = 0.5x_{t-1} + w_t = 0.5(0.5x_{t-2} + w_{t-1}) + w_t$$

$$= 0.25x_{t-2} + 0.5w_{t-1} + w_t$$

$$\cdots\cdots\cdots\cdots$$

$$= (0.5)^r x_{t-r} + \sum_{j=0}^{r-1} (0.5)^j w_{t-j} .$$

If $\lim_{r\to\infty} E\left(\left(x_t - \sum_{j=0}^{r-1}(0.5)^j w_{t-j}\right)^2\right) = \lim_{r\to\infty}(0.5)^{2r} E\left(x_{t-r}^2\right) = 0$, then $\sum_{j=0}^{\infty}(0.5)^j w_{t-j}$ converges to x_t in the mean square sense and so the process can also be written as $x_t = \sum_{j=0}^{\infty}(0.5)^j w_{t-j}$, from which we can easily calculate $\mu_{x_t} = 0$, $\gamma_x(h) = 2^{(2-|h|)}/3$ and $\gamma_x(0) = 4/3$. Hence the process is stationary if $E\left(x_{t-r}^2\right)$ is finite.

Definition 6 A time series $\{x_t\}$ is a linear process if x_t is a linear combination of white noise, w_t, and is written as

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j} , \tag{4}$$

where $\sum_{j=-\infty}^{\infty} \left|\psi_j\right| < \infty$ and μ is a constant.

For the process in (4) it can be shown that

$$E(x_t) = \mu, \quad \forall t \text{, and}$$

$$\gamma_x(h) = E((x_t - \mu)(x_{t+h} - \mu))$$

$$= E\left(\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \, \psi_k \, w_{t-j} \, w_{t+h-k} \right)$$

$$= \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \quad .$$

Clearly the autocovariance function of the process is a function of time lag h only. However, for the process to be stationary the autocovariance function must have finite value at each lag h as shown below.

$$|\gamma_x(h)| = |E((x_t - \mu)(x_{t+t} - \mu))|$$

$$\leq (\mathrm{Var}(x_t)\mathrm{Var}(x_{t+h}))^{1/2} \quad \text{by Cauchy} - \text{Schwarz inequality}$$

$$= \sigma_w^2 \sum_{\text{j=-}\infty}^{\infty} \psi_{\text{j}}^2 = \sigma_w^2 \sum_{\text{j=-}\infty}^{\infty} |\psi_j|^2$$

$$= \sigma_{\text{w}}^2 \left(\left(\sum_{h=-\infty}^{\infty} |\psi_h| \right)^2 - \sum_{h} \underset{\neq}{} \sum_{k} |\psi_h||\psi_k| \right)$$

$$\leq \sigma_{\text{w}}^2 \left(\sum_{h=-\infty}^{\infty} |\psi_h| \right)^2 < \infty \text{ since } \sum_{h=-\infty}^{\infty} |\psi_h| < \infty .$$

If $\psi_j = 0$ for all j < 0, then the process in (4) is said to be causal (see Definition 8 for detail).

2.5 Autoregressive (AR) Process

Shumway and Stoffer (2006) define time series analysis as "the systematic approach by which one goes about answering the mathematical and statistical questions posed by the correlations introduced by the sampling of observations at adjacent points in time." Here by correlation we mean the dependence of the time series observations on one another. This dependence between observations is the most relevant feature that characterizes the dynamics of the underlying system. In time series analysis, there are useful ways of expressing this dependence existing between time series observations. One of these ways is by using some mathematical model that expresses a value of x at time t as a linear combination of its own past values and white noise. This way of representing a time series is referred to as autoregressive form.

Definition 7 A zero mean stationary time series $\{x_t\}$ is an autoregressive process of order p, AR(p), if we can write x_t in the form

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + ... + \phi_p x_{t-p} + w_t , \qquad (5)$$

where ϕ_1, ϕ_2 ,..., ϕ_p are real numbers with $\phi_p \neq 0$ and w_t is white noise.

If $E(x_t) = \mu \neq 0$, then set $y_t = x_t - \mu$, $\forall t$ so that $y_t = w_t + \sum_{j=1}^{p} \phi_j y_{t-j}$.

We can also write (5) in the form $\phi(B)x_t = w_t$, where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - ... - \phi_p B^p$ and B is a backshift operator defined as $B^j x_t = x_{t-j}$, $\forall t, j$. An AR(p) process is also called an AR(p) model. Replacing B by z in $\phi(B)$ gives a polynomial in (6); this is referred to as an AR polynomial.

$$\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j \tag{6}$$

Definition 8 An AR(p) model $\phi(B)x_t = w_t$ is causal if we can write x_t as a linear process

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t \ , \tag{7}$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, $\sum_{j=0}^{\infty} |\psi_j| < \infty$, $\psi_0 = 1$ and $\psi_j = 0, \forall\, j < 0$.

Theorem 1 An AR(p) model is causal if and only if $\phi(z) \neq 0$ for $|z| \leq 1$ (Special case of Property P3.1, Shumway and Stoffer, 2006).

Proof Suppose that $\phi(z) \neq 0$ for $|z| \leq 1$. Then the roots $z_1, z_2, ... z_p$ of $\phi(z) = 0$ lie outside of the unit circle. Assume with out loss of generality that $1 < |z_1| \leq |z_2| \leq ... \leq |z_p|$. Let $|z_1| = 1 + \varepsilon$ for some $\varepsilon > 0$. Then $\phi(z) \neq 0$ for all z such that $|z| < |z_1| = 1 + \varepsilon$. Therefore, $\phi^{-1}(z)$ exists $\forall\, |z| < 1 + \varepsilon$ and has a power series expansion

$$\phi^{-1}(z) = \sum_{j=0}^{\infty} \psi_j z^j, \quad |z| < 1 + \varepsilon \ .$$

Choose a δ such that $0 < \delta < \varepsilon$ and let $z = 1 + \delta$. Then it follows that

$$\phi^{-1}(z) = \phi^{-1}(1+\delta) = \sum_{j=0}^{\infty} \psi_j (1+\delta)^j \ < \ \infty \ .$$

Hence the sequence $\{\psi_j (1+\delta)^j\}$ is bounded. That is, there exists a k > 0 such that $\left|\psi_j (1+\delta)^j\right| < k$, which implies that $\left|\psi_j\right| < k(1+\delta)^{-j}$. From this result, it follows that $\sum_{j=0}^{\infty} \left|\psi_j\right| = k \sum_{j=0}^{\infty} (1+\delta)^{-j} < \infty$. Therefore, $\phi^{-1}(B)$ exists. Premultiplying both sides of the AR model $\phi(B)x_t = w_t$ by $\phi^{-1}(B)$ and setting $\phi^{-1}(B)$ equal to $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$ gives

$$x_t = \phi^{-1}(B)w_t = \psi(B)w_t = \sum_{j=0}^{\infty} \psi_j B^j w_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} \ .$$

Therefore, by definition 8, the AR(p) model is causal.

Suppose that the AR(p) process is causal. That is, suppose x_t has the representation $x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B)w_t$, where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, $\sum_{j=0}^{\infty} \left|\psi_j\right| < \infty$ and $\psi_0 = 1$. Multiplying both sides of $x_t = \psi(B)w_t$ by $\phi(B)$ gives $\phi(B)x_t = \phi(B)\psi(B)w_t$. Comparing this to the AR(p) model $\phi(B)x_t = w_t$, we have $w_t = \phi(B)\psi(B)w_t$, and hence $1 = \phi(B)\psi(B)$. From this result we conclude that $1 = \phi(z)\psi(z)$, $|z| \le 1$ or equivalently $\psi(z) = \dfrac{1}{\phi(z)}$, for $|z| \le 1$.

However, we have $\left|\psi(z)\right| = \left|\sum_{j=0}^{\infty}\psi_j\, z^j\right| \le \sum_{j=0}^{\infty}\left|\psi_j\right|\left|z\right|^j < \sum_{j=0}^{\infty}\left|\psi_j\right| < \infty$ for $|z| \le 1$ and hence, $\left|\phi^{-1}(z)\right| < \infty$, for $|z| \le 1$. This means $\phi(z) \ne 0$ for all $|z| \le 1$. □

2.6 Spectral Density

In time series analysis properties of a phenomenon are studied in terms of its behavior in either the time or frequency domain. In the time domain, future values of a time series are modeled as function of present and past values of the time series. On the other hand, in the frequency domain, the fluctuation of the time series is often modeled in terms of trigonometric sine and cosine functions. Indeed, these trigonometric functions are in turn functions of Fourier frequencies. This section focuses on the description of the function that characterizes the time series in the frequency domain.

Suppose $\{x_t\}$ is a zero mean stationary time series. Then we say that the autocovariance function of the process, $\gamma_x(h)$, is absolutely summable if

$$\sum_{h=-\infty}^{\infty}\left|\gamma_x(h)\right| = \gamma_x(0) + 2\sum_{h=1}^{\infty}\left|\gamma_x(h)\right| < \infty . \tag{8}$$

This condition guarantees the convergence of the function in (9). That is, the function in (9) converges uniformly (see Körner, 2004) if (8) holds.

$$f(\omega) = \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_x(h)\, e^{-i\omega h}, \; -\pi \le \omega \le \pi . \tag{9}$$

Since $\gamma_x(h)=\gamma_x(-h),\ \forall h$ and $e^{-i\omega h}=\cos(h\omega)-i\sin(h\omega)$ we can also derive another form of the function as follows.

$$f(\omega)=\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_x(h)\,e^{-i\omega h}=\frac{1}{2\pi}\left(\sum_{h=-\infty}^{-1}\gamma_x(h)e^{-i\omega h}+\gamma_x(0)+\sum_{h=1}^{\infty}\gamma_x(h)e^{-i\omega h}\right)$$

$$=\frac{1}{2\pi}\left(\sum_{h=1}^{\infty}\gamma_x(-h)e^{i\omega h}+\gamma_x(0)+\sum_{h=1}^{\infty}\gamma_x(h)e^{-i\omega h}\right)$$

$$=\frac{1}{2\pi}\left(\gamma_x(0)+\sum_{h=1}^{\infty}\gamma_x(h)\left(e^{i\omega h}+e^{-i\omega h}\right)\right)$$

$$=\frac{1}{2\pi}\left(\gamma_x(0)+2\sum_{h=1}^{\infty}\gamma_x(h)\cos(h\omega)\right)$$

$$=\frac{1}{2\pi}\left(\sum_{-\infty}^{-1}\gamma_x(-h)\cos(-h\omega)+\gamma_x(0)+\sum_{h=1}^{\infty}\gamma_x(h)\cos(h\omega)\right)$$

$$=\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_x(h)\cos(h\omega)$$

Suppose that condition (8) holds. Then we have

$$\int_{-\pi}^{\pi}f(\omega)e^{i\omega j}d\omega=\int_{-\pi}^{\pi}\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\left(\gamma_x(h)\,e^{-i\omega h}\right)e^{i\omega j}d\omega=\frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma_x(h)\int_{-\pi}^{\pi}e^{i\omega(j-h)}d\omega\,. \qquad (10)$$

Summation and integration are interchanged because of absolute summability of the autocovariance function. The integral in the last expression is simplified as follows.

$$\int_{-\pi}^{\pi} e^{i\omega(j-h)} d\omega \;=\; \begin{cases} 2\pi & if \;\; j = h \\ 0 & if \;\; j \neq h \end{cases} \tag{11}$$

Using (11) in (10) and simplifying gives

$$\int_{-\pi}^{\pi} f(\omega)\, e^{i\omega j} d\omega = \gamma_x(j). \tag{12}$$

From (12) we have $\gamma_x(0) = \int_{-\pi}^{\pi} f(\omega) d\omega$, which means that the area under the function in the interval [-π, π] gives the variance of the process. Note that as stated in Wei (2006) "The term $f(\omega)d\omega$ is the contribution of the variance attributable to the component of the process with frequencies in the interval $(\omega,\ \omega + d\omega)$."

The functions $\gamma_x(h)$ and $f(\omega)$ form Fourier transform pair (see Wei, 2006 for detail). The function f is continuous, non-negative and periodic (with period 2π). In addition, since cosine is an even function, f is also an even function and hence is symmetric around zero. Consequently, values of f are usually determined only at values of ω in the interval [0, π]. Thus, it suffices to consider the graph of such a function only over the interval [0, π].

The condition in (8) holds for any AR(p) process (see Shumway and Stoffer, 2006 for detail). Furthermore, equation (13) presents another way of expressing the function f for AR(p) process (detailed description of this equation is given in Anderson, 1971 and also see Lysne and Tjøstheim, 1987).

$$f(\omega)=\frac{\sigma_{\omega}^{2}}{2\pi}\frac{1}{\left|\phi\left(e^{-i\omega}\right)\right|^{2}}, \text{ where } \phi\left(e^{-i\omega}\right)=1-\sum_{j=1}^{p}\phi_{j}e^{-ij\omega} \quad , \quad -\pi\le\omega\le\pi . \tag{13}$$

Definition 9 Suppose that $\{x_t\}$ is a zero mean stationary time series and condition (8) holds. Then the function in (9) is referred to as the spectral density of the time series $\{x_t\}$.

Example 4 Consider the AR(2) model $x_t = 0.5x_{t-1} - 0.5\,x_{t-2} + w_t$, where w_t is Gaussian white noise with variance one. Then the AR polynomial of the process has roots, $0.5 \pm 0.5\sqrt{7}\,i$, each with modulus $\sqrt{2}$. Hence the process is causal. The spectral density of the process is

$$f(\omega)=\frac{1}{2\pi}\frac{1}{\left|1-.5e^{-i\omega}+.5e^{-2i\omega}\right|^{2}} \quad = \frac{1}{\pi\left[3-3\cos(\omega)+2\cos(2\omega)\right]}, \quad -\pi\le\omega\le\pi .$$

CHAPTER 3

Spectral Peak Frequencies and Arguments of Roots of AR Polynomial of a Causal AR(p) Process

In this chapter we try to investigate the conditions under which the spectral peak frequencies of a causal AR(p) process are either equal or approximately equal to the arguments of roots of the associated AR polynomial for p ≥ 2. More emphasis is given to lower order causal autoregressive processes, say AR(2) and AR(3), to see some pattern so as to draw some generalization for the upper order causal autoregressive processes. All plots in this chapter are based on the assumption of Gaussian white noise with variance one.

3.1 Causal Autoregressive Process of Order 2

In this section we examine properties of roots of the AR polynomial of a causal AR(2) process. We also closely examine the nature of the spectral density of the process and investigate the relationships between the spectral peak frequencies and the arguments of the roots of the AR polynomial of the process.

The AR(2) model is given by $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \omega_t$, where $\{x_t\}$ is a zero mean stationary process, ϕ_1 and ϕ_2 are real numbers with $\phi_2 \neq 0$ and ω_t is white noise with variance σ_w^2. We can also write this model as $\phi(B)x_t = \omega_t$, where $\phi(B) = 1 - \phi_1 B - \phi_2 B^2$

and B is a backshift operator such that $B^j x_t = x_{t-j}$, $\forall t,\ j$. That is B^j maps a value x at time t , j units back in time.

The AR polynomial for the above process is $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$. Solving the quadratic equation $1 - \phi_1 z - \phi_2 z^2 = 0$ for z gives

$$z_1 = \left(-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right) \Big/ 2\phi_2 \text{ and } z_2 = \left(-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right) \Big/ 2\phi_2 \ . \tag{14}$$

The two expressions in (14) are called the roots of the AR polynomial of the process. The causality of the process implies that both of these roots of the AR polynomial lie outside of the unit circle. Consequently,

$$1/z_1 = \left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right) \Big/ 2 \text{ and } 1/z_2 = \left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right) \Big/ 2 \tag{15}$$

exist, from which we obtain $1/z_1 + 1/z_2 = \phi_1$ and $1/z_1 z_2 = -\phi_2$. In addition, because the process is causal it can be shown that $|\phi_1| < 2$ and $|\phi_2| < 1$.

The sign of the quantity $\phi_1^2 + 4\phi_2$ in (14) determines whether the roots are real or complex. That is, if this quantity is negative, then both roots are complex. Otherwise, both roots are real. If both roots are real, then it can be shown that -1 < $1/z_2 \leq 1/z_1$ < 1, from which we obtain the following inequalities: $\begin{cases} \phi_2 + \phi_1 < 1 \\ \phi_2 - \phi_1 < 1 \end{cases}$

Therefore, the following three inequalities are the causal conditions for AR(2) process in terms of the autoregressive coefficients.

$$\begin{cases} |\phi_2| < 1 \\ \phi_2 + \phi_1 < 1 \\ \phi_2 - \phi_1 < 1 \end{cases}$$

The three inequalities form a triangular region given in Figure 1, which is referred to as a causal region for the AR(2) process. All points inside the triangular region, except points on the horizontal axis $(\phi_2 = 0)$, are pairs of values of ϕ_1 and ϕ_2 for which the process is causal. Furthermore, if the point (ϕ_1, ϕ_2) is below the curve determined by the equation $\phi_2 = -0.25\phi_1^2$, then both roots of the AR polynomial are complex. These roots form complex conjugate pair. On the other hand, if the point (ϕ_1, ϕ_2) is on or above the curve $\phi_2 = -0.25\phi_1^2$, then both roots are real. The two real roots are equal if and only if the point (ϕ_1, ϕ_2) lies on the curve $\phi_2 = -0.25\phi_1^2$.

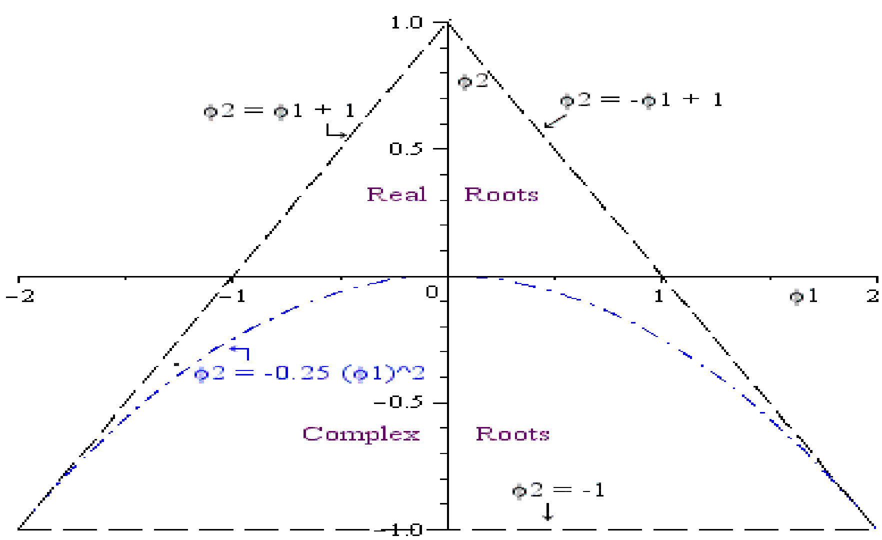

Figure 1: Causal Region for AR(2) Process

Note that we can also write the roots, z_j, in polar coordinates form, $r_j\, e^{i\theta_j}$, where $\theta_j = Arg(z_j)$ is the directed angle measured from the positive x-axis to z_j on the xy complex plane (Pandit and Wu, 1983) and $r_j = |z_j|$ is the modulus of z_j.

Now let us turn our attention to the spectral density of the process. The spectral density of a causal AR(2) process is

$$f(\omega) = \frac{\sigma_w^2}{2\pi\left|1-\phi_1 e^{-i\omega} - \phi_2 e^{-i2\omega}\right|^2}, \quad -\pi \le \omega \le \pi .$$

The form of spectral density that will be used henceforth is given as

$$f(\omega) = \frac{\sigma_w^2}{2\pi\left(1-2\phi_1\cos(\omega)-2\phi_2\cos(2\omega)+2\phi_1\phi_2\cos(\omega)+\phi_1^2+\phi_2^2\right)}$$

$$= \frac{\sigma_w^2}{2\pi\left(-4\phi_2\left(\cos(\omega)-\frac{\phi_1(\phi_2-1)}{4\phi_2}\right)^2+\frac{(4\phi_2+\phi_1^2)(1+\phi_2)^2}{4\phi_2}\right)}. \tag{16}$$

The values of the frequency, ω, that maximize the spectral density are

$$\omega^* = \begin{cases} 0 \\ \pm\pi \\ \pm\arccos\left(\phi_1(\phi_2-1)/4\phi_2\right), & -1 < \phi_1(\phi_2-1)/4\phi_2 < 1 \end{cases}$$

Note that if $\phi_1(\phi_2 - 1)/4\phi_2 = \pm 1$, then $arc\cos(\phi_1(\phi_2 - 1)/4\phi_2) = 0$ or π. On the other hand, if $|\phi_1(\phi_2 - 1)/4\phi_2| < 1$, then $arc\cos(\phi_1(\phi_2 - 1)/4\phi_2)$ lies in the interval $(0, \pi)$. Since ϕ_1 and ϕ_2 are restricted within the triangular region given in Figure 1, we now examine where the inequalities $|\phi_1(\phi_2 - 1)/4\phi_2| < 1$ and $|\phi_1(\phi_2 - 1)/4\phi_2| > 1$ hold within the triangular region.

$$|\phi_1(\phi_2 - 1)/4\phi_2| < 1 \iff -1 < \phi_1(\phi_2 - 1)/4\phi_2 < 1$$

$$\iff \begin{cases} \phi_1/(\phi_1 - 4) < \phi_2 < \phi_1 + 1 \text{ and } \phi_1/(\phi_1 + 4) < \phi_2 < -\phi_1 + 1 \\ \text{OR} \\ -1 < \phi_2 < \phi_1/(\phi_1 - 4) \text{ and } -1 < \phi_2 < \phi_1/(\phi_1 + 4) \end{cases}$$

This means $|\phi_1(\phi_2 - 1)/4\phi_2| < 1$ holds in the regions marked by A and H in Figure 2. While the inequality $|\phi_1(\phi_2 - 1)/4\phi_2| > 1$ holds in regions B, C, D, E, F and G.

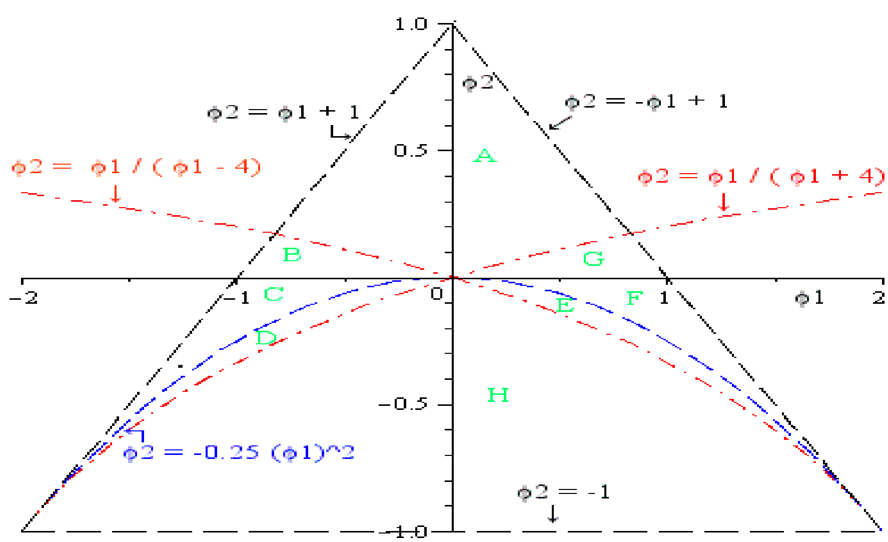

Figure 2: Subsection of Causal Region of AR(2) Process

Now, we compare the spectral peak frequencies and the arguments of the roots of the AR polynomial in each region. Evaluating the second derivative of the spectral density at the values of ω^* gives

$$f''(0) = \frac{-\sigma^2(\phi_1 + 4\phi_2 - \phi_1\phi_2)}{\pi(1+\phi_1^2 + \phi_2^2 - 2\phi_1 - 2\phi_2 + 2\phi_1\phi_2)^2} \tag{17}$$

$$f''(\pm\pi) = \frac{-\sigma^2(-\phi_1 + 4\phi_2 + \phi_1\phi_2)}{\pi(1+\phi_1^2 + \phi_2^2 + 2\phi_1 - 2\phi_2 - 2\phi_1\phi_2)^2} \tag{18}$$

$$f''\left(\pm arc\cos\left(\frac{\phi_1\phi_2 - \phi_1}{4\phi_2}\right)\right) = \frac{-64\phi_2^3\sigma^2\left(\left(\frac{\phi_1\phi_2 - \phi_1}{4\phi_2}\right)^2 - 1\right)}{\pi(1+\phi_2)^4(4\phi_2 + \phi_1^2)^2} \tag{19}$$

We determine whether these values of the spectral density are maximum or minimum depending on the signs of (17), (18) and (19). For instance, using the second derivative test, if (17) has negative sign, then $f(0)$ is a (local) maximum and if (17) has positive sign, then $f(0)$ is a (local) minimum. The spectral peak frequency is the value of ω^* at which f has (local) maximum value.

Region A:

This region is obtained from the intersection of the following two inequalities.

$$\frac{\phi_1}{\phi_1 + 4} < \phi_2 < -\phi_1 + 1 \text{ and } \frac{\phi_1}{\phi_1 - 4} < \phi_2 < \phi_1 + 1$$

$$\Rightarrow \qquad -\phi_1 + \phi_1\phi_2 + 4\phi_2 > 0 \text{ and } \phi_1 - \phi_1\phi_2 + 4\phi_2 > 0$$

$$\Rightarrow \qquad f''(0) < 0 \text{ and } f''(\pm\pi) < 0$$

Furthermore, since $\phi_2 > 0$ it follows that $f''\left(\pm arc\cos\left(\frac{\phi_1\phi_2 - \phi_1}{4\phi_2}\right)\right) > 0$. Therefore, 0 and $\pm\pi$ are the spectral peak frequencies. The frequencies 0 and $\pm\pi$ are absolute and relative spectral peak frequencies, respectively, when $\phi_1 > 0$, and relative and absolute spectral peak frequencies, respectively, when $\phi_1 < 0$.

On the other hand, both roots z_1 and z_2 are real with $z_1 > 0$ and $z_1 < 0$ in this region. Therefore, $\theta_1 = Arg(z_1) = 0$ and $\theta_2 = Arg(z_2) = \pm\pi$. This shows that the spectral peak frequencies are the same as the arguments of the roots of the AR polynomial in the region. Figure 3 is the plot of the spectral density for $\phi_1 = 0.02$ and $\phi_2 = 0.66$ in [0, π].

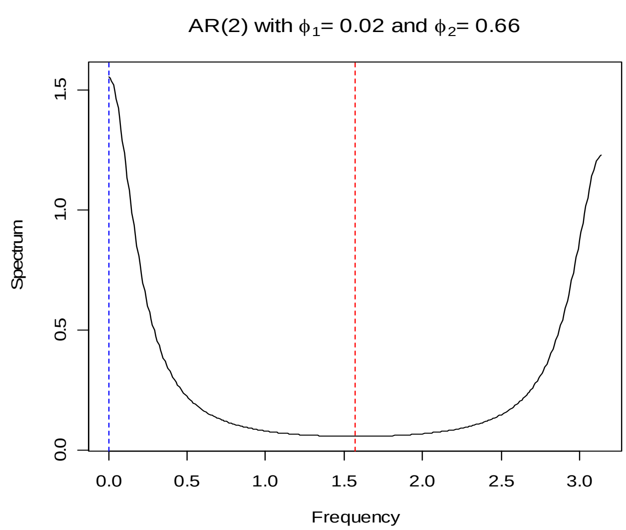

Figure 3: Plot of Spectral Density for Region A

In the plot the blue dotted vertical line indicates the location of the argument of z_1 while the red dotted vertical line indicates the location of $arc\cos(\phi_1(\phi_2-1)/4\phi_2)$ at which the minimum of the spectrum occurs.

Regions B, C and D:

In these regions notice that $arc\cos(\phi_1(\phi_2-1)/4\phi_2)$ does not exist and hence $\omega^*=0$ or $\pm\pi$. In addition, the ineqaulity $\phi_1/(\phi_1+4)<\phi_2<\phi_1/(\phi_1-4)$ holds.

$$\phi_1/(\phi_1+4)<\phi_2<\phi_1/(\phi_1-4) \;\Rightarrow\; -\phi_1+\phi_1\phi_2+4\phi_2>0 \text{ and } \phi_1-\phi_1\phi_2+4\phi_2<0$$

$$\Rightarrow\; f''(0)>0 \text{ and } f''(\pm\pi)<0.$$

Therefore, the spectral peak frequencies are ±π. Moreover, it can be shown that

$$\begin{cases} z_1=\left(\left(-\phi_1+\sqrt{\phi_1^2+4\phi_2}\right)/2\phi_2\right)>0 \\ z_2=\left(\left(-\phi_1-\sqrt{\phi_1^2+4\phi_2}\right)/2\phi_2\right)<0 \end{cases} \Rightarrow \begin{cases} \theta_1=Arg(z_1)=0 \\ \theta_2=Arg(z_2)=\pm\pi \end{cases} \quad \text{in region B}$$

and

$$\begin{cases} z_1=\left(\left(-\phi_1+\sqrt{\phi_1^2+4\phi_2}\right)/2\phi_2\right)<0 \\ z_2=\left(\left(-\phi_1-\sqrt{\phi_1^2+4\phi_2}\right)/2\phi_2\right)<0 \end{cases} \Rightarrow \begin{cases} \theta_1=Arg(z_1)=\pm\pi \\ \theta_2=Arg(z_2)=\pm\pi \end{cases} \quad \text{in region C.}$$

Therefore, the spectral peak frequencies are equal to either $Arg(z_1)$ or $Arg(z_2)$ in region B and C. In addition, since the roots are complex conjugate pair in region D, we can write these roots in polar coordinates as $z_1 = r\,e^{i\theta}$ and $z_2 = r\,e^{-i\theta}$, where $r = |z_1| = |z_2|$, $\theta = Arg(z_1)$ and $-\theta = Arg(z_2)$. Hence it can be shown that $2r\cos(\theta) = -\phi_1/\phi_2$ and $r = \sqrt{-1/\phi_2}$, from which we have $\theta = arc\cos\left(0.5\phi_1\sqrt{-1/\phi_2}\right)$. However, as ϕ_2 tends to $-0.25\phi_1^2$ within the region θ tends to π. Consequently, the condition for the spectral peak frequencies to tend to the arguments of the roots is that ϕ_2 tends to $-0.25\phi_1^2$. That is, the closer the point (ϕ_1, ϕ_2) is to the curve $\phi_2 = -0.25\phi_1^2$ in the region, the closer the spectral peak frequencies are to the arguments of the roots.

Figure 4 represents plots of the spectral densities of AR(2) models for the regions (B, C and D) we have discussed above. Each plot is restricted over the interval [0, π]. The red vertical lines indicate the location of the arguments of the roots of the AR polynomials of the processes. In (B) and (C), the argument of z_2 coincides with the spectral peak frequency. For region D, two plots are considered. In the first plot the argument of z_2 is significantly smaller than the spectral peak frequency while in the second plot the argument of z_2 is close to the spectral peak frequency. In the later the point (ϕ_1, ϕ_2) is very close to the curve formed by the equation $\phi_2 = -0.25\phi_1^2$ in the causal triangular region (Figure 2) while in the former this point is relatively far from the curve.

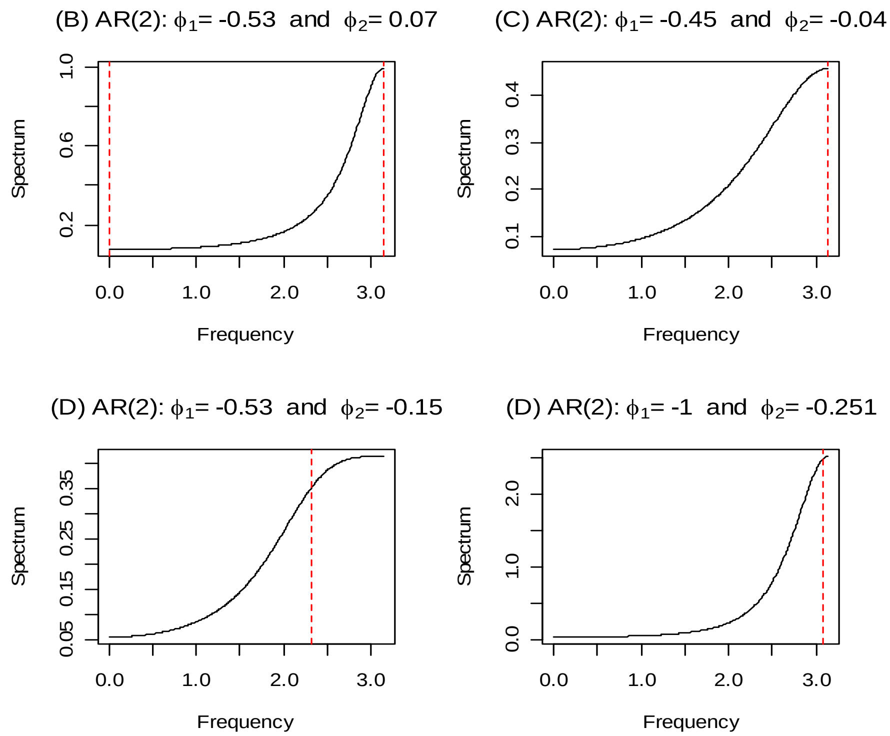

Figure 4: Plots of Spectral Densities of AR(2) Models for Regions B, C and D

Regions E, F and G:

In these regions, $\omega^* = 0 \text{ or } \pm\pi$ since $arc\cos(\phi_1(\phi_2 - 1)/4\phi_2)$ does not exist in these regions. In addition, the inequality $\phi_1/(\phi_1 - 4) < \phi_2 < \phi_1/(\phi_1 + 4)$ holds. However,

$$\phi_1/(\phi_1 - 4) < \phi_2 < \phi_1/(\phi_1 + 4) \;\Rightarrow\; \phi_2 - [\phi_1/(\phi_1 - 4)] > 0 \text{ and } \phi_2 - [\phi_1/(\phi_1 + 4)] < 0$$

$$\Rightarrow\; -\phi_1 + \phi_1\phi_2 + 4\phi_2 < 0 \text{ and } \phi_1 - \phi_1\phi_2 + 4\phi_2 > 0$$

$$\Rightarrow\; f''(0) < 0 \text{ and } f''(\pm\pi) > 0\,.$$

Therefore, the spectral peak of the process occurs at $\omega^* = 0$ in these regions. It is also true that the minimum value of the spectral density occurs at $\omega^* = \pm\pi$.

We now examine what the arguments of the roots of the AR polynomial looks like in these regions and then compare them with the spectral frequencies as follows. In region E, roots are complex conjugate pair. Writing the roots in polar coordinates like as in region D, we can show that $Arg(z_1) = \theta = arc\cos\left(0.5\phi_1\sqrt{-1/\phi_2}\right)$. As ϕ_2 tends to $-0.25\phi_1^2$, $\pm\theta$ tends to 0 (the spectral peak frequency). Hence in this region, the farther the point (ϕ_1, ϕ_2) is from the curve determined by $\phi_2 = -0.25\phi_1^2$, the larger is the difference between the spectral peak frequency and the arguments of the roots, and the closer the point is to the curve the smaller is the difference.

In region F:

$$\begin{cases} z_1 = \left(\left(-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)/2\phi_2\right) > 0 \\ z_2 = \left(\left(-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)/2\phi_2\right) > 0 \end{cases} \Rightarrow \begin{cases} \theta_1 = Arg(z_1) = 0 \\ \theta_2 = Arg(z_2) = 0. \end{cases}$$

In region G:

$$\begin{cases} z_1 = \left(\left(-\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right)/2\phi_2\right) > 0 \\ z_2 = \left(\left(-\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right)/2\phi_2\right) < 0 \end{cases} \Rightarrow \begin{cases} \theta_1 = Arg(z_1) = 0 \\ \theta_2 = Arg(z_2) = \pm\pi. \end{cases}$$

Therefore, the spectral peak frequency is equal to either $Arg(z_1)$ or $Arg(z_2)$ in regions F and G.

The plots of spectral densities of AR(2) models for chosen values of ϕ_1 and ϕ_2 from regions E, F and G are given in Figure 5. In plots the vertical lines are drawn through the arguments of the roots of the AR polynomials of the processes. In (E) when ϕ_2 is near $-0.25\phi_1^2$ within the region (see second plot), the arguments of the roots of the AR polynomial of the process are near 0, the spectral peak frequency. In F and G the argument of z_1 and the spectral peak frequency are the same.

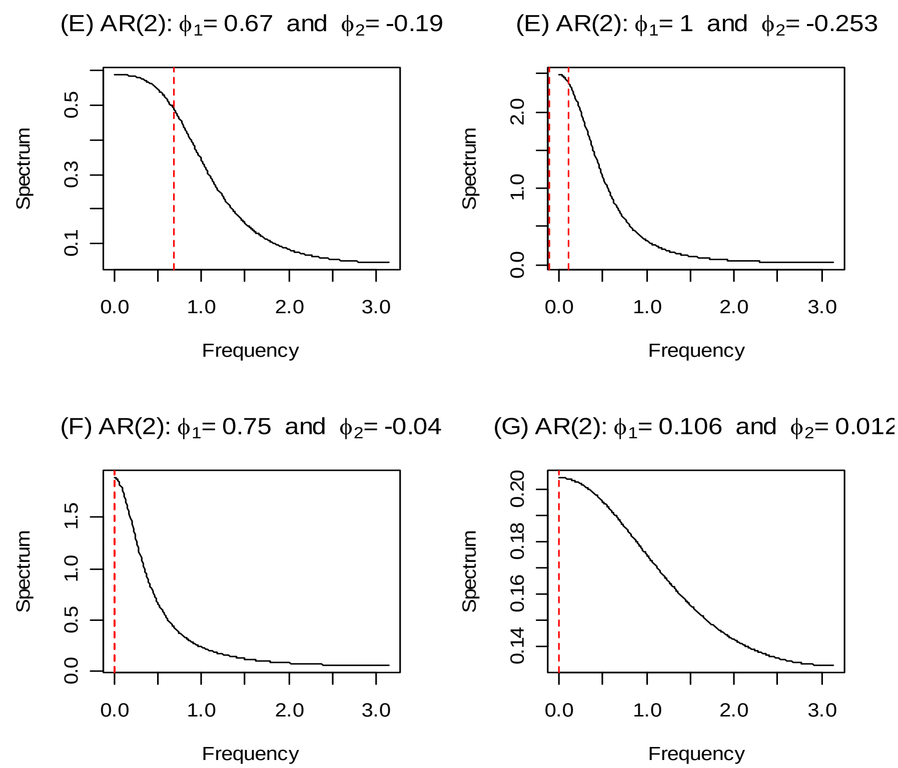

Figure 5: Plots of Spectral Densities of AR(2) models for Regions E, F and G

Region H:

This region is obtained from the intersection of the regions obtained by the inequalities $-1 < \phi_2 < \phi_1/(\phi_1 + 4)$ and $-1 < \phi_2 < \phi_1/(\phi_1 - 4)$. The extreme values of the spectral density occur at $\omega^* = 0, \pm\pi, \pm arc\cos(\phi_1(\phi_2 - 1)/4\phi_2)$ in the region. However,

$$-1<\phi_2<\frac{\phi_1}{\phi_1+4} \text{ and } -1<\phi_2<\frac{\phi_1}{\phi_1-4}$$

$$\Rightarrow \frac{\phi_1}{\phi_1+4}-\phi_2>0 \text{ and } \frac{\phi_1}{\phi_1-4}-\phi_2>0$$

$$\Rightarrow -\phi_1+\phi_1\phi_2+4\phi_2<0 \text{ and } \phi_1-\phi_1\phi_2+4\phi_2<0$$

$$\Rightarrow f''(0)>0 \text{ and } f''(\pm\pi)>0$$

Since $\phi_2<0$ in the region we also have $f''(\pm arc\cos(\phi_1(\phi_2-1)/4\phi_2))<0$. Hence the spectral peaks occur at $\omega^* = \pm arc\cos(\phi_1(\phi_2-1)/4\phi_2)$. The minimum values of the spectral density occur at $\omega^*=0,\pm\pi$.

On the other hand, the roots are complex conjugate pair in this region. Hence, like in region D, writing the roots z_1 and z_2 in polar coordinates it can be shown that $Arg(z_1)=\theta=arc\cos\left(0.5\phi_1\sqrt{-1/\phi_2}\right)$ and $Arg(z_2)=-\theta=-arc\cos\left(0.5\phi_1\sqrt{-1/\phi_2}\right)$.

In addition, the arguments of the roots of the AR polynomial are the same as the spectral peak frequencies when $\phi_1=0$. In this case, the arguments of the roots and the spectral peak frequencies are $\pm\pi/2$. However, when $\phi_1\neq 0$ it is not difficult to show that as ϕ_2 tends to -1, the spectral peak frequencies tend to the arguments of the roots.

Figure 6 represents the plots of the spectral densities of AR(2) models when the autoregressive coefficients are chosen from region H. The plots indicate that the closer

ϕ_2 is to -1 the closer are the arguments of the roots of the AR polynomial to the spectral peak frequencies of the process.

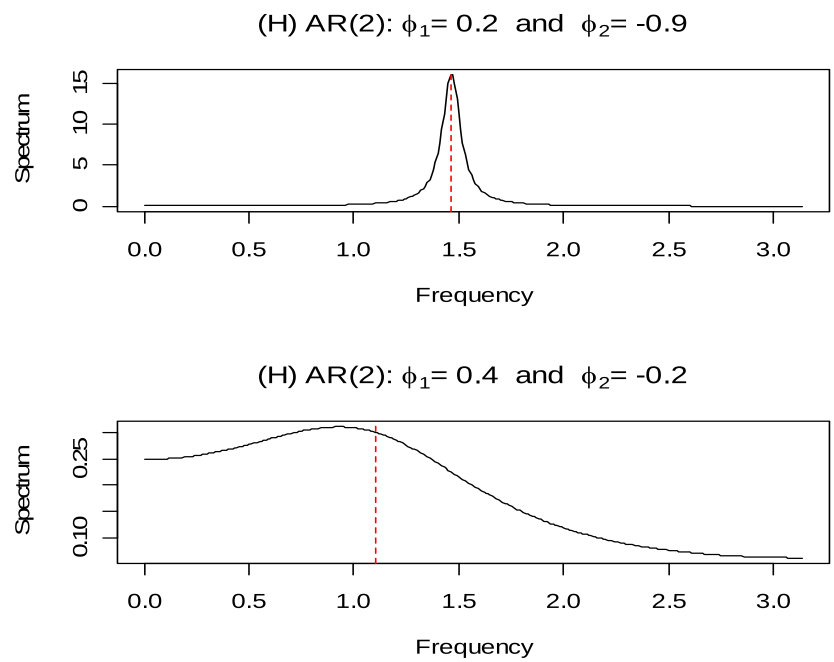

Figure 6: Plots of Spectral Densities of AR(2) Models for Region H

We have shown that the maximum point on the spectral density occurs at a frequency equal to one of the arguments of the roots of the AR polynomial when the roots are real. In addition, when the roots are complex we have seen that the maximum points of the spectral density may occur at or near arguments of the roots of the AR polynomial. To make this more understandable we give the contours of $\theta - \omega^*$ over the interval $[0, \pi]$ in Figure 7, where $\theta = Arg(z_1)$ and both roots are complex. When the roots are real obviously the spectral peak occurs at one of the arguments of the roots and hence this is omitted from the figure. As we can see the figure, $\theta - \omega^* > 0$ when $\phi_1 > 0$,

$\theta - \omega^* < 0$ when $\phi_1 < 0$ and $\theta = \omega^*$ when $\phi_1 = 0$. In addition, the difference is close to zero near $\phi_2 = -1$.

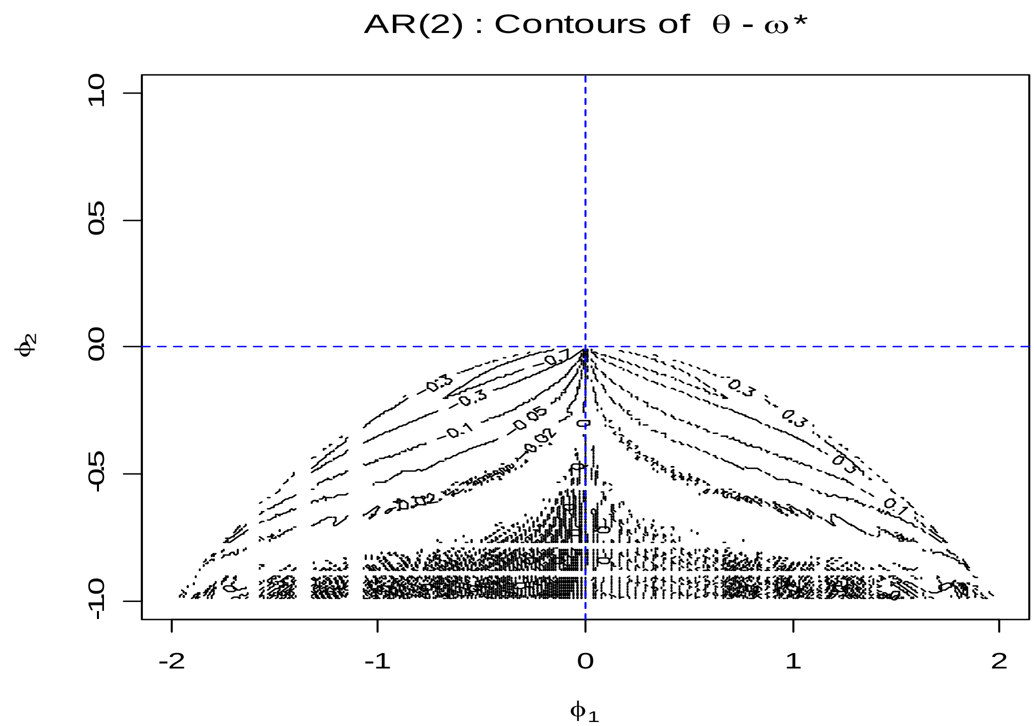

Figure 7 : Plot of Contours of $\theta - \omega^*$, where $\theta = Arg(z_1)$

It is also remarkable to see the relationships between argument of any root of an AR polynomial and the spectral peak frequencies when the root is near the unit circle. Any root, say z, of an AR polynomial can be written in a polar coordinate form $r\,e^{-i\theta}$, where $r = |z|$ and $-\theta = Arg(z)$. As r tends to 1^+, $z = r\,e^{-i\theta}$ tends to $e^{-i\theta}$. Hence $\phi(e^{-i\theta})$ tends to zero since z is the root of $\phi(z) = 0$, from which we conclude that $f(\theta)$ tends to the spectral peak. Since the spectral density is symmetrical about zero it follows that $f(-\theta)$ also tends to the spectral peak. Therefore, $-\theta$ tends to the spectral peak frequency.

The contours of the minimum values of $|z_1|$ and $|z_2|$ are given in Figure 8. As we can see from the figure at least one of the roots is very close to the unit circle when (ϕ_1, ϕ_2) is near the boundaries of the triangular region.

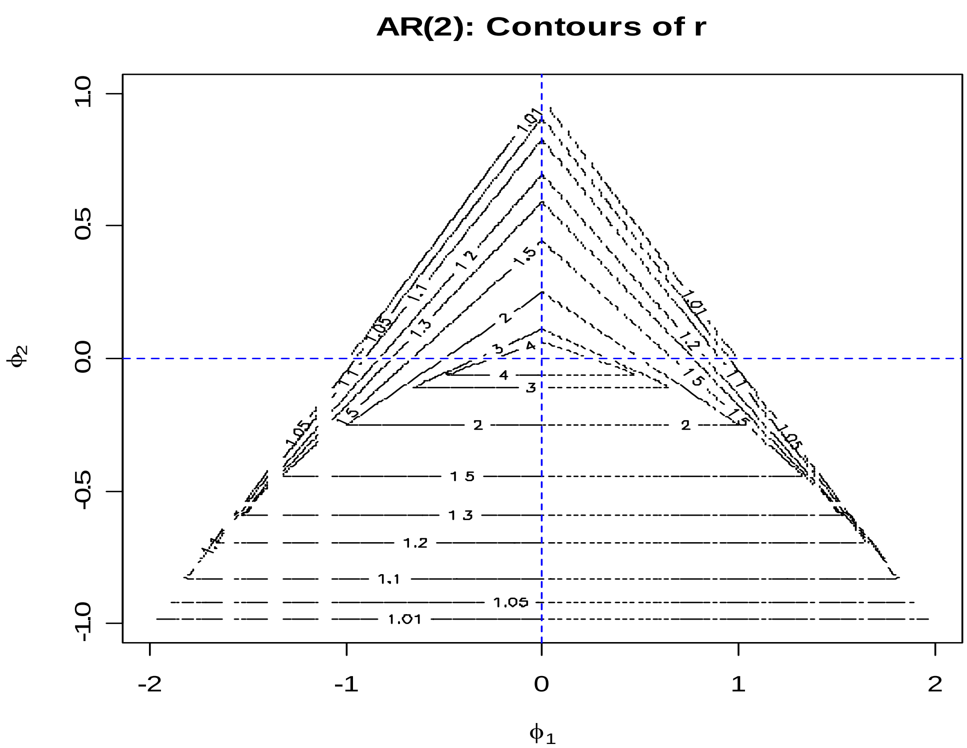

Figure 8: Plot of Contours of $r = \text{Minimum}(|z_1|, |z_2|)$

3.2 Causal Autoregressive Process of Order 3

In this section, the causal conditions for AR(3) process will be presented in terms of the autoregressive coefficients. For given value of ϕ_3, we will examine the nature of the remaining AR coefficients when the process is causal. The relationship between the arguments of the roots of an AR polynomial and the spectral peak frequencies of the process is also investigated.

The AR(3) model is given by $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \phi_3 x_{t-3} + w_t$, where the ϕ_j are real numbers with $\phi_3 \neq 0$ and w_t is white noise with variance σ_w^2. The AR polynomial of the process is $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \phi_3 z^3$. This process is causal if and only if all the roots of $\phi(z) = 0$ lie outside the unit circle by Theorem 1. However, it is possible to construct equivalent conditions based on the autoregressive coefficients. Pandit and Wu (1983) constructed causal conditions for AR(p) process based on the autoregressive coefficients ϕ_j of the process, from which we obtain the causal conditions given in (20) for the AR(3) process.

$$\begin{cases} \phi_1 + \phi_2 + \phi_3 < 1 \\ -\phi_1 + \phi_2 - \phi_3 < 1 \\ |\phi_3| < 1 \\ \phi_3^2 - \phi_1\phi_3 - \phi_2 < 1 \end{cases} \Leftrightarrow \begin{cases} \phi_1 + \phi_2 < 1 - \phi_3 \\ -\phi_1 + \phi_2 < 1 + \phi_3 \\ |\phi_3| < 1 \\ -\phi_3\phi_1 - \phi_2 < 1 - \phi_3^2 \end{cases} \tag{20}$$

By adding the 2nd and 4th inequalities in (20), we get $-\phi_1(1+\phi_3) < (1+\phi_3)(2-\phi_3)$ and dividing both sides of the result by $-(1+\phi_3)$ gives $\phi_1 > -2 + \phi_3$. Similarly, from the

1st and 4th inequalities we get $\phi_1 < 2 + \phi_3$. From these resulting two inequalities we obtain the inequality $-2 + \phi_3 < \phi_1 < 2 + \phi_3$. In addition, the 4th inequality in (20) shows that $\phi_3^2 - \phi_1\phi_3 - 1 < \phi_2$. Moreover, from the 1st and 2nd inequalities we have $\phi_2 < 1 - \phi_1 - \phi_3$ and $\phi_2 < 1 + \phi_1 + \phi_3$. Letting m to be the minimum of $1 - \phi_1 - \phi_3$ and $1 + \phi_1 + \phi_3$, the conditions in (20) are equivalently written as

$$\begin{cases} |\phi_3| < 1 \\ -2 + \phi_3 < \phi_1 < 2 + \phi_3 \\ \phi_3^2 - \phi_1\phi_3 - 1 < \phi_2 < m \end{cases} \tag{21}$$

For any given value of ϕ_3, it can be shown that there exist a $\in$ (-3, -1) and b $\in$ (1, 3) such that $\phi_1 \in$ (a, b). In addition, if $\phi_3 < 0$, then $\phi_2 \in$ (a, 1), b – a = 4, 2 < 1 – a < 4 and (-1, 1) $\subset$ (a, 1) $\subset$ (a, b). In general, if $k = \max\{|a|, b\}$, then $\phi_2 \in (-k, \quad 1)$ with 2 < 1 + k < 4.

For a particular value of ϕ_3, the causal conditions are reduced to a triangular region. For instance, the triangular regions in Figure 9(a) through Figure 9(k) represent the causal regions for AR(3) models for given values of ϕ_3.

Now let us describe the relationships between the spectral peak frequencies and the arguments of the roots of AR polynomial for some given values of ϕ_3. Although $\phi_3 \neq 0$ by definition, if we assume that $\phi_3 = 0$, then the inequalities in (20) given above are reduced to the causal conditions for AR(2) process discussed in Section 3.1. For comparison, the causal region for AR(2) is given in Figure 9(a).

If ϕ_3 is positive, then the region behaves like the triangular regions in Figure 9(b), (d), (f), (g) and (j). In such a case the closer ϕ_3 is to 1, the closer the triangular region is to a line segment with end points (ϕ_1, ϕ_2)= (-1, 1) and (ϕ_1, ϕ_2)= (3, -3). If ϕ_3 is negative, then the region behaves like the triangular regions in Figure 9(c), (e), (g), (i) and (k). The closer ϕ_3 is to -1, the closer is the triangular region to a line segment with end points (ϕ_1, ϕ_2)= (1, 1) and (ϕ_1, ϕ_2)= (-3, -3). The closer ϕ_3 is to ± 1, the closer is the modulus of each root to 1^+.

In Figure 9(e), as (ϕ_1, ϕ_2) tends to the line $\phi_2 - .2\phi_1 = -.96$ in the triangular region, the modulus of each of the complex roots tends to 1^+ and the arguments for these roots also tend to the frequencies that maximize the spectral density.

In Figure 9(f), if ϕ_1 and ϕ_2 are chosen near $\phi_1 + \phi_2 = 0.5$, then the modulus of complex roots significantly deviate from 1 while if they are chosen in the neighborhood of $\phi_2 + .5\phi_1 = -.75$ within the triangular region, then the modulus of each of the complex roots tends to 1^+ and also the argument of these roots tend to the frequencies that maximize the spectral density. In the later case the complex roots are near the unit circle while in the former case the complex roots deviate from the unit circle. That is, the closer the points (ϕ_1, ϕ_2) are to the line $\phi_1 + \phi_2 = 0.5$, the farther the complex roots are from the unit circle.

In Figure 9(h), all the three roots of the AR polynomial are near the unit circle for all (ϕ_1, ϕ_2) in the triangular region. This is due to the fact that the closer ϕ_3 is to

±1, the closer are the roots of the AR polynomial to the unit circle (see Theorem 2 in Section 3.3). Similarly, in Figure 9(i), all the three roots are near the unit circle for all (ϕ_1, ϕ_2) in the triangular region. In this case the arguments of the roots tend to the spectral peak frequencies of the process.

In general, if $\phi_3^2 - \phi_1\phi_3 - \phi_2$ tends to 1^-, then the modulus of at least one of the roots of the AR polynomial tends to 1^+. In this case this root is near the unit circle and the argument of the root tends to the frequency that maximizes the spectral density of the process. In addition, the farther the values of $\phi_3^2 - \phi_1\phi_3 - \phi_2$ are from 1^-, the farther the roots are from the unit circle and hence the larger the difference between the arguments of the roots and the spectral peak frequencies.

On the other hand, if both ϕ_1 and ϕ_2 are zero, then the arguments of the roots of the AR polynomial are exactly the same as the spectral peak frequencies.

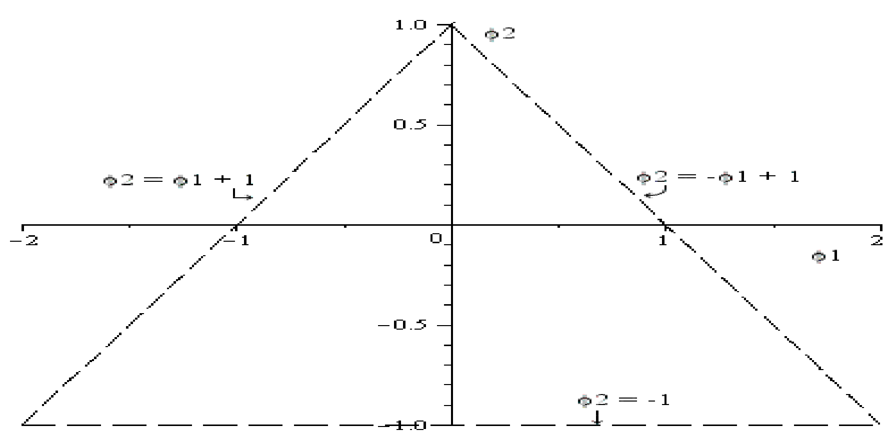

Figure 9(a): Causal region for AR(2) process

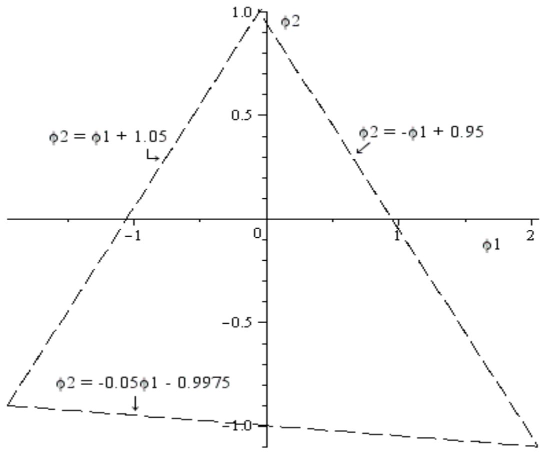

Figure 9(b): Causal Region for AR(3)

with $\phi_3 = 0.05$

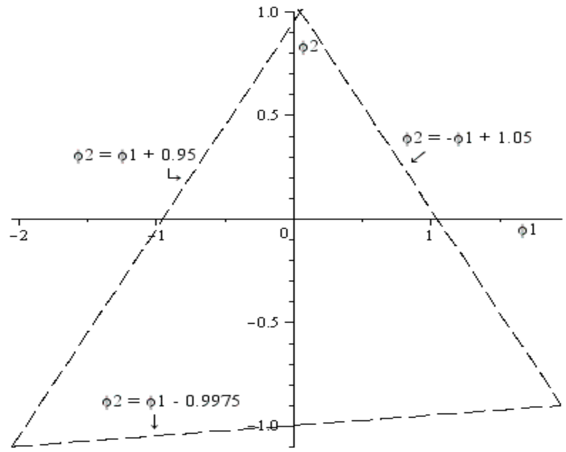

Figure 9(c): Causal Region for AR(3)

with $\phi_3 = -0.05$

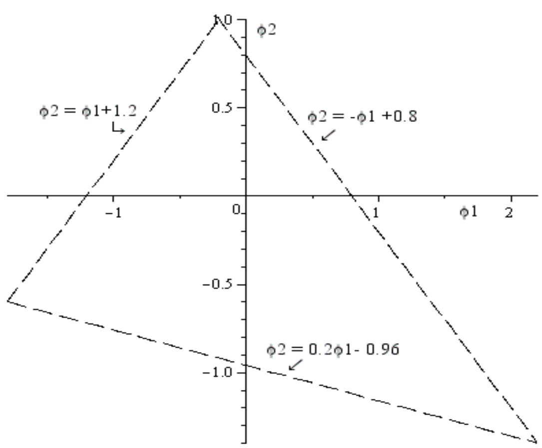

Figure 9(d): Causal Region for AR(3)

with $\phi_3 = 0.2$

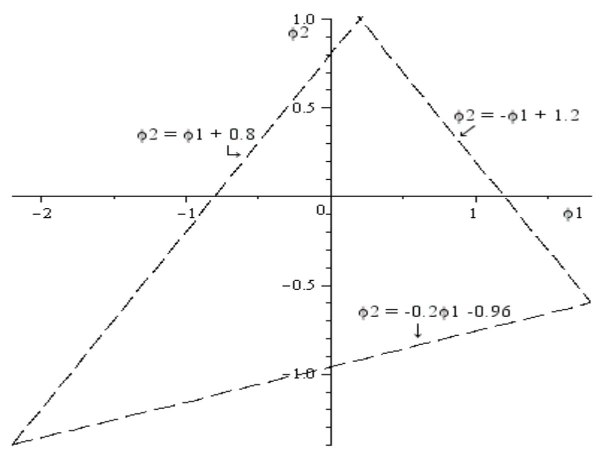

Figure 9(e): Causal Region for AR(3)

with $\phi_3 = -0.2$

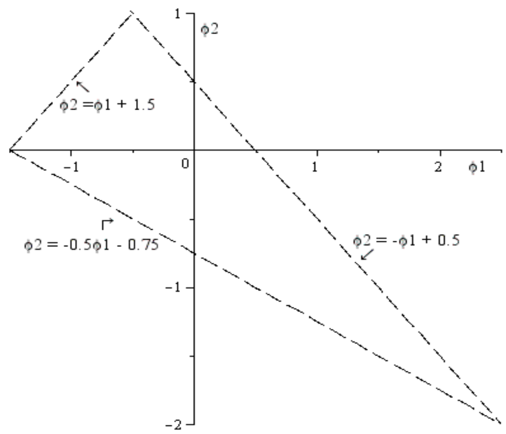

Figure 9(f): Causal Region for AR(3)

with $\phi_3 = .5$

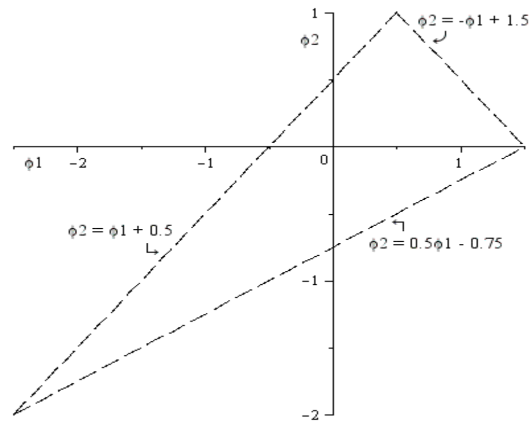

Figure 9(g): Causal Region for AR(3)

with $\phi_3 = -.5$

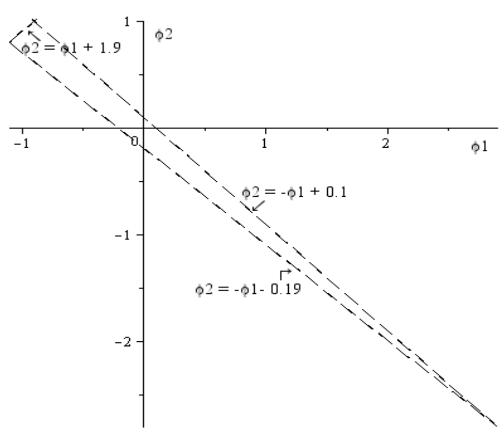

Figure 9(h): Causal Region for AR(3)

with $\phi_3 = 0.9$

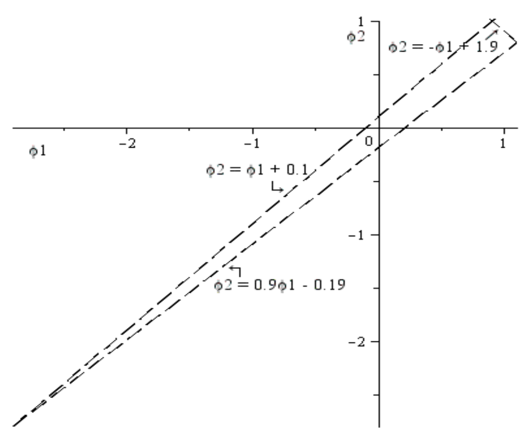

Figure 9(i): Causal Region for AR(3)

with $\phi_3 = -0.9$

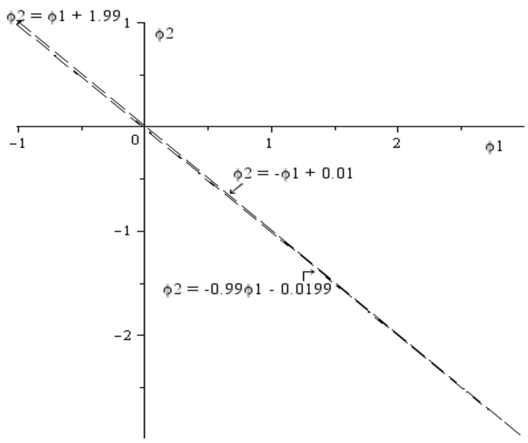

Figure 9(j): Causal Region for AR(3)

with $\phi_3 = 0.99$

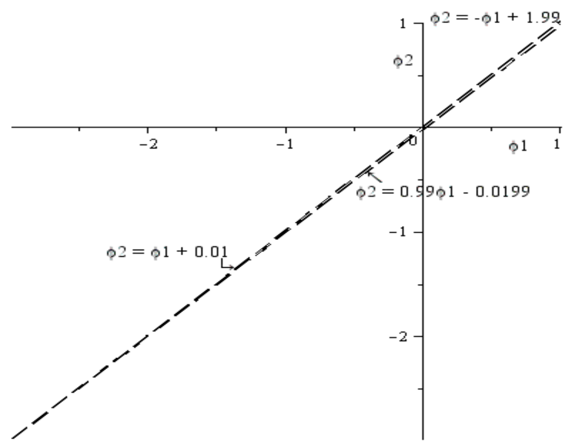

Figure 9(k): Causal Region for AR(3)

with $\phi_3 = -0.99$

Suppose that the AR polynomial has two complex roots and one real root. Then the argument of the real root equal to the spectral peak frequency if its modulus is smaller than that of the complex roots. In this case spectral peak frequency is either 0 or $\pm\pi$. The arguments of the complex roots of the AR polynomial tend the spectral peak frequencies if these roots are very close to the unit circle and the real root is far from the unit circle.

Example 5 Figure 10 is the plot of the spectral density of AR(3) with autoregressive coefficients $\phi_1 = -1.37$, $\phi_2 = -0.37$, and $\phi_3 = 0.36$ over the interval $[0, \pi]$. In the figure, the dashed line, which coincides with one of the dotted lines, corresponds to the spectral peak frequency (approximately 2.62). On the other hand, the AR polynomial of the process has complex conjugate pair roots, $-0.869 \pm 0.498i$ (to three decimal places), with modulus and arguments approximately equal to 1 and ± 2.62, respectively, and a real root, 2.766 (to three decimal places), with modulus approximately equal to 2.766 and argument equal to 0. In the figure, the dotted vertical lines correspond to arguments of the real root and the complex root $-0.869 + 0.498i$. It is not difficult to see that the spectral peak frequency is very close to the argument of this complex root.

Example 6 Figure 11 shows the plot of the spectral density of a causal AR(3) process with autoregressive coefficients $\phi_1 = .15$, $\phi_2 = .53$, and $\phi_3 = .28$. The AR polynomial of the process has roots 1.019 and $-1.456 \pm 1.176i$ (to three decimal places). In the figure, the dotted lines correspond to the arguments, 0 and 2.46, of the AR polynomial roots, 1.019 and $-1.456 + 1.176i$, respectively. The modulus of the complex conjugate pair roots is approximately 1.9. The frequencies that maximize the spectral density in the interval $[0, \pi]$ are approximately 0 and 2.66. The dashed line corresponds to the frequency 2.66. Observe that there is a noticeable difference between 2.46 and 2.66. The significance of this difference is due to the fact that the closeness of the modulus of the real root to one overwhelms the effect of the complex root at frequency 2.66. Notice that the spectral peak frequency, 0, and the argument of the real root are the same.

In general, if the modulus of one of the roots of the AR polynomial is near one, then the argument of this root and the spectral peak frequency are either the same or approximately the same.

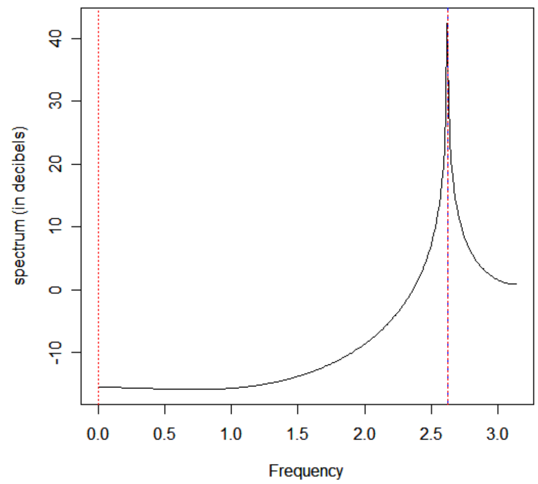

Figure 10: Plot of Spectral Density of AR(3); $\phi_1 = -1.37$, $\phi_2 = -.37$, $\phi_3 = .36$

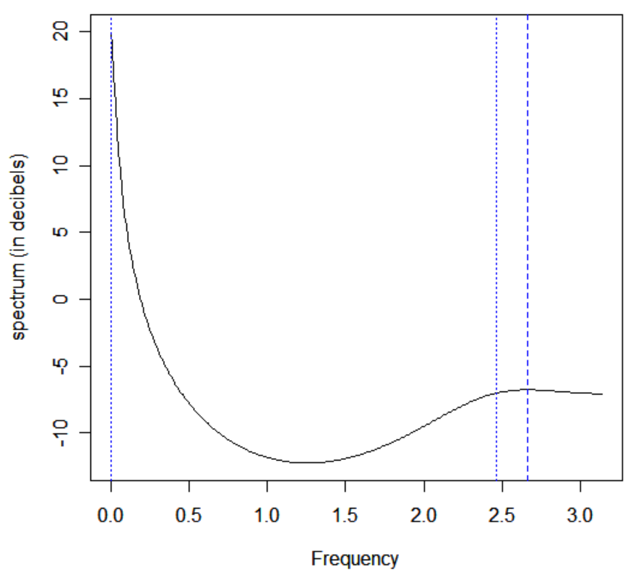

Figure 11:Plot of Spectral Density of AR(3); $\phi_1 = .15$, $\phi_2 = .53$, $\phi_3 = .28$

3.3 Causal Autoregressive Process of Order P for P ≥ 4

In this section we start our discussion by proving Theorem 2. This theorem gives the relationship between ϕ_p and the modulus of each root of the AR polynomial of a causal AR(p) process. In the two preceding sections we discussed about conditions under which the spectral peak frequencies are equal to or tend to the arguments of the roots of the AR polynomial for causal AR(2) and AR(3) processes. A sufficient condition for the spectral peak frequencies to be equal to or approximately equal to the arguments of the roots of the AR polynomial was that the roots are near the unit circle. We prove this statement as Theorem 3 for any causal AR(p) process in this section.

Theorem 2 Let $x_t = w_t + \sum_{j=1}^{p} \phi_j x_{t-j}$ be a causal AR(p) process with AR polynomial $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j$. Let z_j , $j = 1, 2, ..., p$ be roots of $\phi(z)$. Then $\left|\phi_p\right|$ tends 1^- if and only if $\left|z_j\right|$ tends 1^+ , $\forall\, j$.

Proof : Given a causal AR(p) model $x_t = w_t + \sum_{j=1}^{p} \phi_j x_{t-j}$, where w_t is white noise with variance σ_w^2. Consider the AR polynomial, $\phi(z) = 1 - \sum_{j=1}^{p} \phi_j z^j$, of the process. Then we can factorize $\phi(z)$ as $\left(1 - z_1^{-1} z\right)\left(1 - z_2^{-1} z\right) ... \left(1 - z_p^{-1} z\right)$, where z_j are roots of an AR polynomial for $j = 1, 2, \ldots, p$. By expanding this product we also get $(-1)^p z_1^{-1} z_2^{-1} \ldots z_p^{-1}$ as a coefficient of z^p and comparing this to the coefficient of the last term of the AR polynomial gives $(-1)^p z_1^{-1} z_2^{-1} \ldots z_p^{-1} = -\phi_p$. Then from the causality of the process we get $|\phi_p|$ tends 1^- if and only if $|z_j|$ tends 1^+, $\forall\, j$. □

The above theorem indicates that if ϕ_p is near one or negative one, then all the roots of the AR polynomial are near the unit circle and conversely, if all the roots are near the unit circle, then ϕ_p is near one or negative one. However, if at least one of the roots is far from the unit circle, then ϕ_p deviates from one or negative one accordingly.

In Section 3.1 we have seen that if one of the roots of an AR polynomial of a causal AR(2) process is near the unit circle, then the argument of that root tends to the spectral peak frequency. We now prove a related theorem for a causal AR(p) process.

Theorem 3 Let $z_j = r_j e^{-i\theta_j}$ be a root of the AR polynomial of a causal AR(p) process. Then there exists a spectral peak frequency ω^* such that

$$|z_j| \to 1^+ \Rightarrow \begin{cases} \theta_j = \omega^* = 0 \text{ or } \pm\pi & \text{if } z_j \text{ is a real root} \\ \theta_j - \omega^* \to 0 & \text{if } z_j \text{ is a complex root.} \end{cases}$$

Proof : Suppose that $z_j = r_j e^{-i\theta_j}$ is one of the roots of the AR polynomial, where $r_j = |z_j|$ and $-\pi \le \theta_j \le \pi$. If r_j tends to 1^+, it follows that z_j tends to $e^{-i\theta_j}$. Therefore, $\phi\left(e^{-i\theta_j}\right)$ tends to $\phi(z_j) = 0$, which implies that $f(\theta_j) = \sigma_w^2 \Big/ \left(2\pi \left|\phi\left(e^{-i\theta_j}\right)\right|^2\right)$ is in the neighborhood of one of the peaks of the spectral density. Hence θ_j is near ω^*, where ω^* is one of the spectral peak frequencies. From the symmetric properties of the spectral density it follows that $-\theta_j$ tends to $-\omega^*$, the spectral peak frequency. If the root is real, then obviously $\theta_j = 0$ or $\pm\pi$ and hence is equal to the spectral peak frequency. □

The converse of Theorem 3 is not true. To demonstrate this, suppose that $\phi_j = 0$, $\forall\, j < p$ and $\phi_p \neq 0$. Then the spectral density of the process is the function

$$f(\omega) = \frac{\sigma_w^2}{2\pi\left(1+\phi_p^2 - 2\phi_p \cos(p\omega)\right)}\ , \quad -\pi \le \omega \le \pi\,.$$

It can be shown that Extreme values of the function occur at frequencies, $\omega^* = j\pi/p$, where $j = 1, 2, 3, \ldots, p$. The spectral peak frequencies can easily be identified from these frequencies depending on the sign of ϕ_p and whether p is odd or even.

If p is odd, the spectral peak frequencies are given by

$$\omega^* = \begin{cases} 0, \pm\dfrac{2\pi}{p}, \ \ldots\ , \pm\dfrac{(p-1)\pi}{p} & \text{if } \phi_p > 0 \\ \pm\dfrac{\pi}{p}, \pm\dfrac{3\pi}{p}, \ldots, \pm\dfrac{(p-2)}{p}\pi, \pi & \text{if } \phi_p < 0 \end{cases}$$

If p is even, the spectral peak frequencies are given by

$$\omega^* = \begin{cases} 0, \pm\dfrac{2\pi}{p}, \ \ldots\ , \pm\dfrac{(p-2)}{p}\pi, \pi & \text{if } \phi_p > 0 \\ \pm\dfrac{\pi}{p}, \pm\dfrac{3\pi}{p}, \ldots, \pm\dfrac{(p-1)\pi}{p} & \text{if } \phi_p < 0 \end{cases}$$

On the other hand, the AR polynomial of the process is $\phi(z) = 1 - \phi_p z^p$. If p is odd, then this polynomial has $p-1$ complex roots and one real root. The arguments of these roots are

$$\theta = \begin{cases} 0, \pm\dfrac{2\pi}{p}, \ \dots \ , \pm\dfrac{(p-1)\pi}{p} & \text{if } \phi_p > 0 \\ \pm\dfrac{\pi}{p}, \pm\dfrac{3\pi}{p}, \dots, \pm\dfrac{(p-2)}{p}\pi, \pi & \text{if } \phi_p < 0 \end{cases}$$

If p is even, then the polynomial has p complex roots when $\phi_p < 0$, and two real and $p-2$ complex roots when $\phi_p > 0$. The arguments of these roots are given by

$$\theta = \begin{cases} 0, \pm\dfrac{2\pi}{p}, \ \dots \ , \pm\dfrac{(p-2)}{p}\pi, \pi & \text{if } \phi_p > 0 \\ \pm\dfrac{\pi}{p}, \pm\dfrac{3\pi}{p}, \dots, \pm\dfrac{(p-1)\pi}{p} & \text{if } \phi_p < 0 \end{cases}$$

Consequently, $\omega^* = \theta$ for any causal AR(p) process with $\phi_j = 0, \forall j < p$ and $\phi_p \neq 0$. However, if $\phi_j = 0, \forall j < p$ and ϕ_p is in the neighborhood of zero, then the modulus of each root of the AR polynomial is significantly larger than one, which contradicts the converse of Theorem 3. In fact, all the roots of such a case have the same modulus.

CHAPTER 4

Conclusion

In this chapter some important points of the study are discussed by way of summarizing results from previous chapters. Applications of Theorem 2 and Theorem 3 are presented.

An autoregressive process of order two is causal when the coefficients of the AR polynomial satisfy the conditions $\phi_2 + \phi_1 < 1$, $\phi_2 - \phi_1 < 1$ and $|\phi_2| < 1$. Under these conditions we found that the spectral peak frequencies tend to or are equal to the arguments of the roots of the AR polynomial when $|\phi_2|$ is near 1. The spectral peak frequency is equal to at least one of the arguments of the roots of the AR polynomial when both roots are real. We also found that when ϕ_1 is zero the arguments of the roots are exactly equal to the spectral peak frequencies of the process, regardless of the value of ϕ_2. In addition, the arguments of the complex roots of the AR polynomial are found to be approximately equal to the spectral peak frequencies when ϕ_1 is in the neighborhood of zero.

For an AR(2) process if $\phi_2 \in D \cup E$ (see Figure 2 for the regions) such that ϕ_2 is not near -1, then the spectral peak frequencies significantly deviate from the arguments of the roots. In this case we found that the plot of the spectral density is either strictly increasing or strictly decreasing over the interval [0, π] and hence the peaks of the plot of the spectral density occur at frequencies 0 or π. However, the arguments of the roots of the AR polynomial of the process lie in the interval $(0, \pi)$. The closer ϕ_2 is to $-.25\phi_1^2$ the smaller are the differences between the arguments of the roots of the AR polynomial and the spectral peak frequencies. In addition, the closer ϕ_2 is to $\phi_1/(\phi_1 - 4)$ or $\phi_1/(\phi_1 + 4)$ the larger are the differences between the arguments of the roots of the AR polynomial and the spectral peak frequencies.

An AR(3) process is causal whenever the coefficients of the AR polynomial of the process satisfy the conditions in (20). For a causal AR(3) process the argument of any root of the process is approximately or exactly equal to one of the spectral peak frequencies when $|\phi_3|$ tends to 1^-. In addition, each spectral peak frequency is equal to one of the arguments of the roots of the AR polynomial of the process when both ϕ_1 and ϕ_2 are zero.

We found in Section 3.3 that the argument of any root of the AR polynomial of a causal AR(p) process is equal to one of the spectral peak frequencies when $\phi_j = 0$, $\forall\, j < p$. In addition, we found out that for a causal AR(p) process, $p \geq 2$, the argument of any root of the AR polynomial of the process tends to or is equal to one of the spectral peak frequencies when its modulus tends to 1^+ or equivalently when $|\phi_p|$ tends to 1^-. This has the following remarkable application.

Suppose that we are interested in the spectrum of a causal AR(p) process that has only one peak in the interval [0, π], say at a frequency $\pi/3$. Then using the results given above we can find a process with this property. Clearly the AR polynomial of the process has complex conjugate pair roots. Therefore, the order of the process could be greater than or equal to two. If ϕ_p is very close -1, then the process could be of order two, in which case the order of the process is equal to twice the number of peaks in the interval. The process could also be of order three. This happens when the two complex roots are located near the unit circle and the real root is located far from the unit circle, in which the complex roots overwhelm the effect of the real root on the spectral density. There could be other possibilities, which we can interpret similarly.

Now we demonstrate the case where the process is of order two. Theorem 2 indicates that $|\phi_2|$ tends to 1^- is equivalent to $|z_j|$ tends to 1^+, $\forall\, j$. It follows from Theorem 3 that the argument of one of roots of the AR polynomial tend to $\pi/3$ and that of the other tends to $-\pi/3$. Suppose that $r = |z_1| = |z_2| \approx 1.01$ so that $\theta = Arg(z_1)$ tends to $\pi/3$ and $-\theta = Arg(z_2)$ tends to $-\pi/3$. In addition, writing the roots in polar coordinates and using them in the AR polynomial gives

$$\begin{cases} 1-\phi_1\left(r\,e^{i\theta}\right)-\phi_2\left(r\,e^{i\theta}\right)^2 = 0 \\ 1-\phi_1\left(r\,e^{-i\theta}\right)-\phi_2\left(r\,e^{-i\theta}\right)^2 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} 1-\phi_1\left(1.01e^{i\frac{\pi}{3}}\right)-\phi_2\left(1.01e^{i\frac{\pi}{3}}\right)^2 \approx 0 \\ 1-\phi_1\left(1.01e^{-i\frac{\pi}{3}}\right)-\phi_2\left(1.01e^{-i\frac{\pi}{3}}\right)^2 \approx 0 \end{cases}$$

Solving this system of equations gives $\begin{cases} \phi_1 \approx 0.99 \\ \phi_2 \approx -0.98 \end{cases}$

Hence we can use these values of ϕ_1 and ϕ_2 to approximate the causal AR(2) model with the given properties by the model $x_t = 0.99\,x_{t-1} - 0.98\,x_{t-2} + w_t$, where $\{x_t\}$ is a zero mean stationary process and w_t is white noise. The roots of the AR polynomial of this process are approximately $0.505 \pm 0.875i$ (rounded to three decimal places). The arguments of these roots are approximately equal to ± 1.047 (rounded to three decimal places).

We can also simulate the plot of spectral density of the above process. Figure 12 represents the plot of the spectral density of the process when the white noise process is Gaussian. Notice that the spectral peak frequency $\pi/3 \approx 1.047$ is approximately the same as the argument of one of the roots.

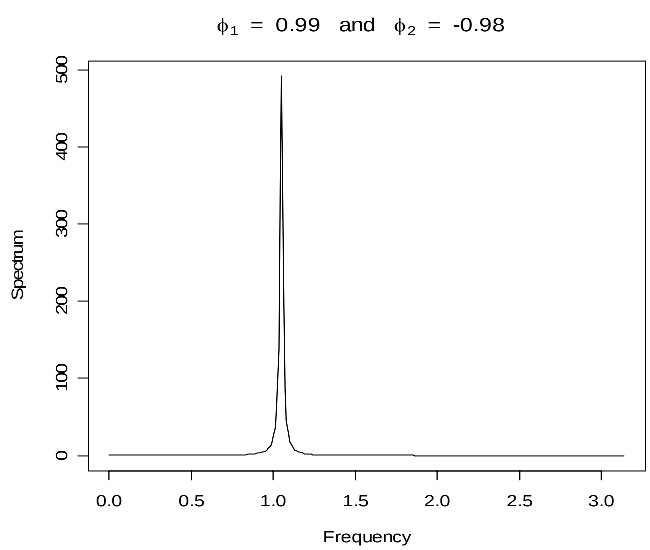

Figure 12 : Plot of Spectral Density of $x_t = 0.99\,x_{t-1} - 0.98\,x_{t-2} + w_t$

A process of order three is also possible. We demonstrate this as follows. Here we force the complex roots to be located near the unit circle and the real root to be located far from the unit circle so that the arguments of the complex roots tend to $\pm\pi/3$. To accomplish this let us choose the modulus of the real root to be approximately equal to 10 and that of the complex roots to be approximately equal to 1.01. Assume that the real root is positive so that its argument is 0. Then using these information in the AR polynomial of the AR(3) process shows that

$$\begin{cases} 1-\phi_1\left(r_1\, e^{i\theta_1}\right)-\phi_2\left(r_1\, e^{i\theta_1}\right)^2-\phi_3\left(r_1\, e^{i\theta_1}\right)^3=0 \\ 1-\phi_1\left(r_2\, e^{-i\theta_2}\right)-\phi_2\left(r_2\, e^{-i\theta_2}\right)^2-\phi_3\left(r_2\, e^{-i\theta_2}\right)^3=0 \\ 1-\phi_1\left(r_3\, e^{i0}\right)-\phi_2\left(r_3\, e^{i0}\right)^2-\phi_3\left(r_3\, e^{i0}\right)^3=0 \end{cases}$$

$$\Rightarrow \begin{cases} 1-\phi_1\left(1.01e^{i\frac{\pi}{3}}\right)-\phi_2\left(1.01e^{i\frac{\pi}{3}}\right)^2-\phi_3\left(1.01e^{i\frac{\pi}{3}}\right)^3\approx 0 \\ 1-\phi_1\left(1.01e^{-i\frac{\pi}{3}}\right)-\phi_2\left(1.01e^{-i\frac{\pi}{3}}\right)^2-\phi_3\left(1.01e^{-i\frac{\pi}{3}}\right)^3\approx 0 \\ 1-\phi_1\left(10e^{i0}\right)-\phi_2\left(10e^{i0}\right)^2-\phi_3\left(10e^{i0}\right)^3\approx 0 \end{cases}$$

$$\Rightarrow \begin{cases} \phi_1\approx 1.090 \\ \phi_2\approx -1.079 \\ \phi_3\approx 0.098 \end{cases}$$

Therefore, we can approximate the causal AR(3) model with the given properties by

$$x_t = 1.090\,x_{t-1} - 1.079\,x_{t-2} + 0.098\,x_{t-3} + w_t\,,$$

where $\{x_t\}$ is a zero mean stationary process and w_t is white noise.

Under the assumption that the white noise process is Gaussian, the plot of the spectral density of the process is approximated by the simulated plot given in Figure 13.

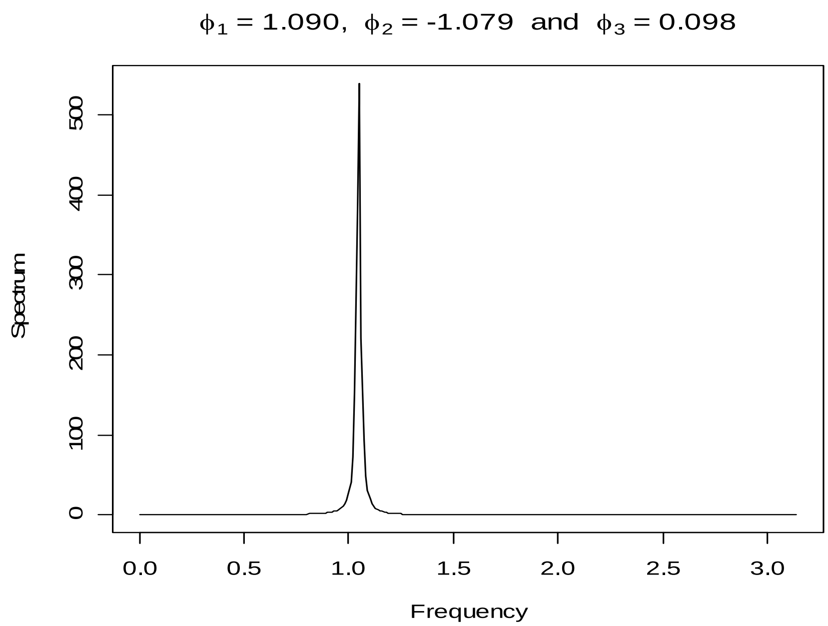

Figure 13 : Plot of Spectral Density of $x_t = 1.090\,x_{t-1} - 1.079\,x_{t-2} + 0.098\,x_{t-3} + w_t$

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