M133a  Ex. 5    Spring 2017    R.K. Dodd

Due: (01) 04/12/17, (02) 04/13/17.

Attempt all the questions. All problems must have work attached on the scratch sheets.

NAME:

1. Find the solutions to the following IVPs:

   (a) \( y'' + 4y' + 3y = e^{-t}, \quad y(0) = 1, y'(0) = 2 \)
   \[ \frac{1}{3} + \frac{1}{2} \]

   (b) \( y'' + y = \sin x, \quad y(\pi) = 0, y'(\pi) = 0 \)
   \[ \frac{1}{4} + \frac{1}{2} \]

   (c) \( y'' + y' + y = 3x^2 - 2x + 1, \quad y(0) = 1, y'(0) = 0 \)
   \[ \frac{1}{2} + \frac{1}{2} \]

   Ans.

2. For the following equations determine whether the method of undetermined coefficients can be used to obtain the particular solution. If possible write out the form of the particular solution.

   (a) \( y'' + 2y' - y = x^{-1} \sin x \)
   \[ \frac{1}{2} \]

   (b) \( y'' + y = x^3 \cos x + x e^{5x} \)
   \[ \frac{1}{2} \]

   (c) \( y'' + y' + y = \tan x \)
   \[ \frac{1}{2} \]

   Ans.

3. Determine whether the following differential operators are linear or nonlinear.

   (a) \( R[y] := y'' + x^2 y \)
   \[ \frac{1}{2} \]

   (b) \( S[y] := y'' + 2y' \sin x + 6y \)
   \[ \frac{1}{2} \]

   (c) \( T[y] := y'' + y^2 \)
   \[ \frac{1}{2} \]

   Ans.

4. Establish whether each of the following sets of functions are linearly independent or linearly dependent, (a) \( \{ \sin 2x, e^{3x}, x^2 \} \), (b) \( \{ \cos 3x, \sin 2x, \tan x \} \),

   (c) \( \{ x^3, x^5, \cos x, x^3 - 5x^5 \} \), (d) \( \{ 1, \tan x, \sec^2 x \} \).

   Ans.

5. Suppose that the populations of two different animals \( P \) and \( Q \) in a given habitat are respectively \( p(t) \) and \( q(t) \). The \( Q \) animals have a growth rate which is proportional to their population, because the habitat has a plentiful supply of
their food. The $P$ animals survive by preying only upon the $Q$ animals and will die out exponentially if the $Q$ population is very small. Assume that the rate at which the $P$ population grows is proportional to the product of both their populations at any given time. Show that a possible model for this situation is provided by the equations

$$\frac{dq}{dt} = aq - bpq, \quad \frac{dp}{dt} = -cp + dpq, \quad (\ast)$$

where $a, b, c$ and $d$ are positive constants. The constant solutions of these equations are called equilibrium solutions. Thus for example if $q = q^*, p = p^*$ is an equilibrium solution of the system then $dq^*/dt = 0, dp^*/dt = 0$.

(a) Find the possible equilibrium solutions for the animal populations.

(b) Recall that the average of a function $f(t)$ over a time $T$ is defined by

$$\frac{1}{T} \int_0^T f(t) dt.$$ 

It can be shown that the populations of the animals vary periodically, that is there exists a time $T$ such that $p(t + T) = p(t)$ and $q(t + T) = q(t)$. Use the equations $(\ast)$ to show that the average values of the $P, Q$ populations over a period $T$ is one of the possible equilibrium solutions for the $P$ and $Q$ populations.

Note: You do not need to know $T$ to be able to do this problem.

Ans.
1. (a) \( y'' + 4y' + 3y = e^t, \ y(0) = 1, \ y'(0) = 2. \)

Hom. eqn.: \( y'' + 4y' + 3y = 0 \)

Char. eqn.: \( \lambda^2 + 4\lambda + 3 = 0, \ (\lambda + 3)(\lambda + 1) = 0 \)

Gen. soln.: \( y_h = c_1 e^{-3t} + c_2 e^{-t} \)

Particular soln.: \( y_p = A t e^t \)

\[ A \left( -2te^t + t e^t \right) + 4(e^t - te^t) + 3te^{2t} = e^t \]

\[ A(-t + 1) + 3A = 0 \]

A valid, satisfies: \( e^t - 2A + 4A = 1 \)

\( A = \frac{1}{2} \) so \( y_p = \frac{1}{2} te^t \)

Gen. soln. of inhom. eqn.: \( y = y_h + y_p \)

\[ y = c_1 e^{-3t} + c_2 e^{-t} + \frac{1}{2} te^t \]

Imposed data: \( 1 = y(0) = c_1 + c_2, \ 2 = y'(0) = -3c_1 - c_2 + \frac{1}{2} \)

Soln. of IVP: \( y = -\frac{5}{2} e^{-3t} + \frac{1}{2} e^{-t} + \frac{1}{4} te^t \)

(b) \( y'' + y = \sin x, \ y(\pi) = 0, \ y'(\pi) = 0 \)

Hom. eqn.: \( y'' + y = 0, \) char. eqn.: \( \lambda^2 + 1 = 0, \ \lambda = \pm i \)

Complex soln.: \( e^{ix} = \cos x + i \sin x \)

Gen. soln. of hom. eqn.: \( y_h = c_1 \cos x + c_2 \sin x \)

Particular soln. of inhom. eqn.: \( y_p = A \cos x + B \sin x \)

\[ A \left( -2 \sin x - x \cos x + x \cos x \right) + B \left( 2 \cos x - x \sin x + x \sin x \right) = \sin x \]

\( (\sin x) \cdot 2A = 1 \) \( \cos x \) \( 2B = 0 \)
\[ y_p = -\frac{1}{2} x \cos x \]

General solution of inhomogeneous equation:

\[ y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x \]

Impose data:

\[
\begin{align*}
0 &= y(\pi) = c_1 \cos \pi + c_2 \sin \pi - \frac{1}{2} \pi \cos \pi = -c_1 + \frac{1}{2} \\
0 &= y'(\pi) = -c_1 \sin \pi + c_2 \cos \pi - \frac{1}{2} \pi \sin \pi + \frac{1}{2} \pi \cos \pi \\
&\text{Solve, get } c_1 = \frac{1}{2}, c_2 = \frac{1}{2} \\
\end{align*}
\]

Specific solution of IVP:

\[ y = \frac{1}{2} \pi \cos x + \frac{1}{2} \sin x - \frac{1}{2} x \cos x \]

\[(c)\] \[ y'' + y' + y = 3x^2 - 2x + 1, \quad y(0) = 1, \quad y'(0) = 0 \]

Homogeneous equation:

\[ y'' + y' + y = 0 \]

Characteristic equation:

\[ \lambda^2 + \lambda + 1 = 0 \]

Complex roots:

\[ \lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \]

General solution:

\[ y_h = \frac{e^{\frac{-x}{2}}}{2} \left( c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) \]

Particular solution:

\[ y_p = A x^2 + B x + C \]

\[(x)\] \[ A x^2 + B + C = 3x^2 - 2x + 1 \]

\[(x)\] \[ A + B = -2 \]

\[(x)\] \[ 2A + B + C = 1 \]

Solve:

\[ A = 3, \quad B = -2 - 2A = -8, \quad C = 1 - 2A - B = 3 \]

Specific solution:

\[ y = \frac{e^{\frac{-x}{2}}}{2} \left( c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x \right) + 3x^2 - 8x + 3 \]
Impose data

\[ 1 = y(0) = c_1 + 3 \]
\[ 0 = y'(0) = \frac{1}{2} c_1 + \frac{5}{2} c_2 - 8 \text{ or } c_2 = \frac{14}{5} \]

Solution of IVP

\[ f = -2 e^{5x} \cos \frac{5}{2} x + \frac{14}{5} e^{5x} \sin \frac{5}{2} x + 3 x^2 - 8 x + 3 \]

If the eqn. has the form \( \frac{dy}{dx} = f \), assume that \( y_p \) is a linear combination of the linearly independent frs. in the list

\[ \{ f, f', f'', \ldots \} \]

This will work provided there is only a finite no. of such frs.

(a) Form a list \( \{ x, \sin x \} \) and its derivatives

\[ \{ x, \sin x, x \cos x, x^2 \sin x, x^2 \cos x, \ldots, x^3 \sin x, x^3 \cos x, \ldots \} \]

There are an infinite no. of lin. ind. frs. in the list so the Method of Undetermined Coeffs. is not applicable.

(b) \( f(x) = x^3 \cos x + x e^5 x \)

The lin. ind. frs.

\[ \{ x^3 \cos x, x e^5 x, x^3 \sin x, e^5 x, x^3 \cos x, x^3 \sin x, \cos x, \sin x^3 \} \]

Thus the method works and a possible particular soln. has the form

\[ y_p = (a x^3 + A_2 x^2 + A_3 x + A_4) \cos x + (B x^3 + B_2 x^2 + B_3 x + B_4) \sin x + (C x + C_2) e^{5x} \]
(1) In this case the list is
\[ \{ \tan x, \sec^2 x, \sec^2 x \tan x, \ldots \} \]

since \( \sec^2 x = 1 + \tan^2 x \), then \( \sec^2 x \tan x = (1 + \tan^2 x) \tan x \)
on or \( \sec^2 x \tan x = \tan x + \tan^3 x \), etc. Thus the
list of independent \( \{1, \tan x, \tan^2 x, \tan^3 x, \ldots \} \)
does this is an infinite list, the method will also work.

3 (a) \[ R[Eg] = y'' y' + y' \]
\[ R[c, f + g] = (c f'' + g'') (c f' + g' ) + x (c f + g) \]
where
\[ c, R[f] ] + c, R[g] = c, f'' f' + c, x f + g' g' + x g g' \]
These two expressions must be identical if
\( R[f] \) is a linear operator. However, the first
expression contains the term \( c, f'' f' \) which is absent from the
second expression. The operator is therefore nonlinear.

(b) \[ S[c, f + g] = c, f'' + g'' + 2(c f' + g') \sin x + 6(c, f + g) \]
\[ = c, (f'' + 2 \theta f' \sin x + 6f) + \epsilon_2 (g'' + 2g' \sin x + 6g) \]
\[ = c, S[f] + \epsilon_2 S[g] \]
the operator \( S[f] \) is linear.

c) \[ T[Eg] = y''' + y' \]
\[ T[c, f + g] = c, f''' + g''' + (c, f' + g') \]
\[ T[f] + \epsilon_2 T[g] = c, (f''' + f') + \epsilon_2 (g''' + g') \]
is absent from the second. The operator is therefore non-linear.

\[ c_1 \sin 2x + c_2 e^{3x} + c_3 x^2 = 0 \] (*)

This has to be true for every \( x \in \mathbb{R} \). Thus if \( x = 0 \), we get \( 0 + c_2 + 0 = 0 \) so \( c_2 = 0 \). The expression (*) reduces to

\[ c_1 \sin 2x + c_3 x^2 = 0 \]

Put \( x = \frac{\pi}{2} \) and we get \( 0 + c_3 (\frac{\pi}{2})^2 = 0 \) so \( c_3 = 0 \).

We are left with \( c_1 \sin 2x = 0 \) so \( c_1 = 0 \).

Since \( c_1 = 0, c_2 = 0, c_3 = 0 \) the fns. are lin. ind.

b) Form \( c_1 \cos 3x + c_2 \sin 2x + c_3 \tan x = 0 \)

Put \( x = 0 \), expression reduces to

\[ c_2 \sin 2x + c_3 \tan x = 0 \]

Lots of ways to handle this.

**Method 1**

\[ x = \frac{\pi}{2} \]

\[ c_2 \sin \frac{\pi}{2} + c_3 \tan \frac{\pi}{2} = 0 \]

\[ c_2 + c_3 = 0 \]

**Method 2**

\[ c_2 \cos x \sin x + c_3 \tan x = 0 \]

Since \( \sin x \) is non-zero for:

\[ x = 0 \]

\[ 2c_2 + c_3 = 0 \]

\[ 2c_2 + c_3 = 0 \]

\[ c_2 + \frac{1}{2} c_3 = 0 \]

\[ c_2 + c_3 = 0 \]

\[ c_2 = 0, c_3 = 0 \]

Thus \( 0, 0 \)

are lin. ind.
Thus, \( c_1 = 0 \), \( c_2 = 0 \), \( c_3 = 0 \) and the fns. are lin. ind.

(c) \( \{ x^3, x^5, \cos x, x^3 - 5x^5 \} \)

Form \( c_1 x^3 + c_2 x^5 + c_3 \cos x + c_4 (x^3 - 5x^5) = 0 \) \( \cdots \) \( (x) \)

**Method 1** Observe that \( x^3 - 5x^5 = (x^3) - 5(x^5) \) and so the fns. are lin. dep. (lin. dep. does not require all the fns. to be involved!)

**Method 2** Put \( x = 0 \) in \( (x) \) \[ c_3 = 0 \]

Relation becomes

\[ c_1 x^3 + c_2 x^5 + c_3 (x^3 - 5x^5) = 0 \]

Since \( x^3 \) is not the zero fn., we can divide by \( x^3 \)

\[ c_1 + c_2 x^2 + c_3 (1 - 5x^2) = 0 \]

Put \( x = 0 \) \[ c_1 + c_2 = 0 \]

Which gives \[ c_2 x^2 - 5c_4 = 0 \]

Since \( x^2 \) is not the zero fn., divide by it \[ c_2 - 5c_4 = 0 \]

Thus the fns. are linearly dependent with \( c_1 = -c_4 \) and \( c_2 = 5c_4 \) and \( c_3 = 0 \).

(d) \( \{ 1, \tan x, \sec^2 x \} \)

Form \( c_1 + c_2 \tan x + c_3 \sec^2 x = 0 \) \( \cdots \) \( (x) \)

Put \( x = 0 \) \[ c_1 + c_3 = 0 \]

diff. the expression \( (x) \)

\[ c_2 \sec^2 x + 2c_3 \sec x \tan x = 0 \]

\( \sec^2 x \) is not the zero fn., divide by it \[ c_2 + 2c_3 \tan x = 0 \]

Put \( x = 0 \) \[ c_2 = 0 \] and so also \[ c_3 = 0 \]
Thus \( c_1 = 0, c_2 = 0, c_3 = 0 \) and the fns. are lin. inde.

5. \( \frac{dq}{dt} = aq \quad a > 0 \) from the given info.

However there is an interaction with the P animals which decreases the population growth rate according to \( s \) some multiple \( f(p) \) so the actual eqn. for the rate of change of the \( Q \) pop. is \( \frac{dq}{dt} = aq - bpq \).

The \( P \) population feeds on the \( Q \) pop. so a first estimate for the rate of change of the \( P \) population is \( \frac{dp}{dt} = -cp \).

However this is modified by the presence of the \( Q \) population so the final eqn. is

\[
\frac{dp}{dt} = -cp + dpq
\]

giving the system of eqns.

\[
\begin{align*}
\frac{dq}{dt} &= aq - bpq \quad \text{(x)} \\
\frac{dp}{dt} &= -cp + dpq
\end{align*}
\]

Where \( a, b, c, d \) are positive consns. For an equilibrium solution \( p = p^*, q = q^* \) where \( \frac{dp^*}{dt} = 0 \), \( \frac{dq^*}{dt} = 0 \), i.e. they are consn. Subs. \( p = p^*, q = q^* \) into \( \text{(x)} \) gives

\[
q^*(a - bp^*) = 0 \quad 0 = t^*(-c + dq^*)
\]

So the possible consns. are \( p^* = 0, q^* = 0 \) or \( p^* = \frac{q^*}{a}, q^* = \frac{c}{d} \).
(b) Since the populations are T-periodic:
\[ p(t + T) = p(t), \quad q(t + T) = q(t) \]
for all \( t \) and so
\[ \frac{1}{T} \int_0^T p(t) \, dt = \frac{1}{bT} \left( \int_0^T \frac{dq}{dt} + a \int_0^T dt \right) \]
But \( q(T) = q(0) \) so \( \ln(q(T)) - \ln(q(0)) = 0 \) and
\[ \frac{1}{T} \int_0^T q(t) \, dt = \frac{a}{b} \]
Similarly, \[ \frac{1}{T} \int_0^T p(t) \, dt = \frac{c}{a} \]
Thus, the average values of the \( p, q \) populations over a period \( T \) coincides with the equilibrium populations \( p = \frac{a}{b}, q = \frac{c}{a} \).