Householder idea 2: reflection to zero out all but the first component of a given vector $z$

Given any vector $z$ we will construct a Householder transformation $U$ so that $\tilde{z} = Uz$ has all the components of $\tilde{z}$ equal to zero, except for the first component of $\tilde{z}$. We do this by constructing the "mirror" (plane in $\mathbb{R}^3$) such that the reflection of $z$ through $\Pi$ lies along the $-e_1$ axis, where $e_1$ is the vector that is all zero except that the first component is 1.

Construction:

- Let $\sigma = \text{sign}(z_1)||z||$, where, if $z_1 \geq 0$, then $\text{sign}(z_1) = 1$ and, if $z_1 < 0$, then $\text{sign}(z_1) = -1$.
- Let $\hat{v} = z + \sigma e_1$ and $u = \hat{v} / ||\hat{v}||$
- Let $\Pi$ be the plane perpendicular to $u$ (and therefore $U = (I - 2uu^T)$).

Here is a picture

![Diagram of Householder transformation]

To show that the image of $z$ lies along the negative $e_1$ axis, we need to show that angle $d$, that is the angle of "incidence" equals the angle of "reflection." We can do this as follows:

1. $\angle b = \angle a$. Proof:
   Consider parallelogram 1234 with sides $v$ and $\sigma e_1$ and diagonal $z + \sigma e_1$. This is a rhombus since we choose $\sigma$ so that $||z|| = ||\sigma e_1||$. Therefore, by the geometry of a rhombus $\angle b = \angle a$.

2. $\angle b + \angle c = 90^\circ$ follows since $u$ is perpendicular to $\Pi$.

3. $\angle a + \angle b + \angle c + \angle d = 180^\circ$ since the angle between $-\sigma e_1$ and $\sigma e_1$ is $180^\circ$.

4. $\angle a + \angle d = 90^\circ$, subtracting 2 from 3.

5. So $\angle d = 90^\circ - \angle a$ (by 4) and $\angle c = 90^\circ - \angle b$ (by 2). Since, by 1, $\angle b = \angle a$, it follows that $\angle c = \angle d$, as desired.