## Gaussian Elimination with Partial Pivoting Example

Apply Gaussian elimination with partial pivoting to $A = \begin{pmatrix} 1 & 2 & -4 & 3 \\ 2 & 5 & -6 & 10 \\ -2 & -7 & 3 & -21 \\ 2 & 8 & 15 & 38 \end{pmatrix}$ and solve $Ax = b$ for $b = \begin{pmatrix} 0 \\ 9 \\ -28 \\ 42 \end{pmatrix}$.

**Solution:**

Apply Gaussian elimination with partial pivoting to $A$ using the compact storage mode where the multipliers (= elements of L) are stored in $A$ in the locations of $A$ that are to be made zero.The elements of $L$ are in red. Note that when one interchanges rows of the current $A$, one must also interchange rows of the current $L$. For example, in the step at the third arrow, below, we switch the second and fourth rows of $A_2$ and $L_2$. No row interchanges are required for $A_3$, so we have skipped pivoting for $A_3$.

$$\begin{array}{c|rrrrcc|rrrrcc|rrrrc}
\lambda & & A_1 & & & & \lambda & & P_1 & A_1 & & & \lambda & L_2 & \text{and} & A_2 & & \\
1 & 1 & 2 & -4 & 3 & & 2 & 2 & 5 & -6 & 10 & & 2 & 2 & 5 & -6 & 10 & \\
2 & 2 & 5 & -6 & 10 & \to & 1 & 1 & 2 & -4 & 3 & \to & 1 & 1/2 & -1/2 & -1 & -2 & \to \\
3 & -2 & -7 & 3 & -21 & & 3 & -2 & -7 & 3 & -21 & & 3 & -1 & -2 & -3 & -11 & \\
4 & 2 & 8 & -5 & 38 & & 4 & 2 & 8 & -5 & 38 & & 4 & 1 & 3 & 1 & 28 &
\end{array}$$

$$\begin{array}{c|rrrrcc|rrrrcc|rrrr}
\lambda & P_2L_2 & \text{and} & P_2 & A_2 & & \lambda & L_3 & \text{and} & A_3 & & & \lambda & L_4 & \text{and} & A_4 & \\
2 & 2 & 5 & -6 & 10 & & 2 & 2 & 5 & -6 & 10 & & 2 & 2 & 5 & -6 & 10 \\
4 & 1 & 3 & 1 & 28 & \to & 4 & 1 & 3 & 1 & 28 & \to & 4 & 1 & 3 & 1 & 28 \\
3 & -1 & -2 & -3 & -11 & & 3 & -1 & -2/3 & -7/3 & 23/3 & & 3 & -1 & -2/3 & -7/3 & 23/3 \\
1 & 1/2 & -1/2 & -1 & -2 & & 1 & 1/2 & -1/6 & -5/6 & 8/3 & & 1 & 1/2 & -1/6 & 5/14 & -1/14
\end{array}$$

So $U = A_4 = \begin{pmatrix} 2 & 5 & -6 & 10 \\ 0 & 3 & 1 & 28 \\ 0 & 0 & -7/3 & 23/3 \\ 0 & 0 & 0 & -1/14 \end{pmatrix}$ and $L = L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & -2/3 & 1 & 0 \\ 1/2 & -1/6 & 5/14 & 1 \end{pmatrix}$.

P will have a one in columns 2, 4 , 3 and 1, in that order, (see $\lambda$ ) so $P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$. The permutation vector, p, describing the matrix P is p = [2 4 3 1]. $PA = A(p,:) = \begin{pmatrix} 2 & 5 & -6 & 10 \\ 2 & 8 & 15 & 38 \\ -2 & -7 & 3 & -21 \\ 1 & 2 & -4 & 3 \end{pmatrix}$.

Here $A(p,:)$ uses Matlab notation. $LU = PA$ can be checked.

To solve $Ax = b$ note that $PAx = Pb = \widehat{b}$ or $LUx = \widehat{b}$. Therefore we can solve $Ax = b$ in three steps:

(1) Let $\widehat{b} = Pb = \begin{pmatrix} 9 \\ 42 \\ -28 \\ 0 \end{pmatrix}$.

(2) Solve $Lw = \widehat{b}$ using forward substitution. We get $w = \begin{pmatrix} 9 \\ 33 \\ 3 \\ 1/14 \end{pmatrix}$.

(3) Solve $Ux = w$ using back substitution. We get $x = \begin{pmatrix} 3 \\ 1 \\ 2 \\ 1 \end{pmatrix}$.