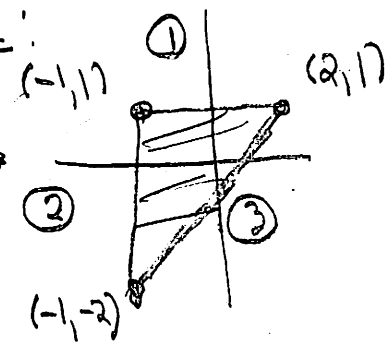
Example Global Extrema for $f(x,y)$

Problem Find Global extrema of $z = f(x,y) = x^2+2xy+3y^2$ over triangle with vertices $(-1,1)$, $(2,1)$ & $(-1,-2)$

Picture:

domain $D \longrightarrow$



on Boundary ①: $y=1$ so

$f(x,y) = f(x,1) = x^2+2x+3, \; -1 \le x \le 2$

on Boundary ②: $x=-1$ so

$f(x,y) = f(-1,y) = 1-2y+3y^2, \; -2 \le y \le 1$

on Boundary ③:

line through $(-1,-2)$ & $(2,1)$ is

$$\frac{y-(-2)}{x-(-1)} = \frac{1-(-2)}{2-(-1)} = \frac{3}{3} = 1$$

or $y+2 = x+1$ or $y = x-1$ so

$f(x,y) = x^2+2x(x-1)+3(x-1)^2$
$\quad = 6x^2-8x+3, \; -1 \le x \le 2$

⓪ critical points in D

$f_x = 2x+2y = 0$

$f_y = 2x+6y = 0$

$\Rightarrow 4y = 0 \Rightarrow y=0 \Rightarrow x=0$

$(0,0)$ is critical & in D

Boundary ① find extrema of:

$g(x) = x^2+2x+3, \; -1 \le x \le 2$

$0 = g'(x) = 2x+2 \Rightarrow x=-1$

candidates for extrema

$x=-1$ critical & $x=-1,2$ endpoints

or $(-1,1)$, $(2,1)$

Boundary ③ find extrema of

$k(x) = 6x^2-8x+3, \; -1 \le x \le 2$

$0 = k'(x) = 12x-8 = 0 \Rightarrow x = \frac{2}{3}$

candidates for extrema

$x = \frac{2}{3}$ & $x=-1,2$ endpoints

or $\left(\frac{2}{3}, -\frac{1}{3}\right)$ $(-1,-2)$, $(2,1)$

using $y = x-1$

Boundary ② find extrema of:

$h(y) = 1-2y+3y^2, \; -2 \le y \le 1$

$0 = h'(y) = -2+6y \Rightarrow y = \frac{1}{3}$

candidates for extrema

$y=\frac{1}{3}$ critical & $y=-2,1$ endpoints

or $\left(-1, \frac{1}{3}\right)$, $(-1,-2)$ & $(-1,1)$

Summary table of potential extrema

Case	x	y	f(x,y)	
⓪	0	0	0	← min
①	-1	1	2	
	2	1	11	
②	-1	$\frac{1}{3}$	$\frac{2}{3}$	
	-1	-2	17	← max
③	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	

so global min is 0 at $(0,0)$ $\Longleftarrow$

global max is 17 at $(-1,-2)$

Final Answer