(14.8)

Lagrange Multiplier Example

max $f(x,y,z) = 3x - y - 3z$ $\leftarrow$ objective function

such that $g(x,y,z) = x+y-z=0$ ① & $h(x,y,z) = x^2+2z^2=1$ ② $\leftarrow$ constraints

$\nabla f = \lambda \nabla g + \omega \nabla h$ ($\leftarrow$ Lagrange multiplier equation)

or

$\langle 3,-1,-3 \rangle = \lambda \langle 1,1,-1 \rangle + \omega \langle 2x, 0, 4z \rangle$

or $3 = \lambda + 2\omega x$, $-1 = \lambda + 0$, $-3 = -\lambda + 4\omega z$

$\lambda = -1$ ④

$3 = -1 + 2\omega x$ $\qquad$ $-3 = 1 + 4\omega z$

$\omega x = 2$ $\qquad$ $\omega z = -1$

③ $x = \frac{2}{\omega}$ $\qquad$ ⑤ $z = -\frac{1}{\omega}$

Now we solve ①–⑤:

②+③+⑤ $\Rightarrow$ $\left(\frac{2}{\omega}\right)^2 + 2\left(-\frac{1}{\omega}\right)^2 = 1 \Rightarrow 2^2 + 2 = \omega^2$, $\omega^2 = 6$

$\Rightarrow \omega = \pm\sqrt{6}$

So $x = \frac{2}{\pm\sqrt{6}} = \pm\frac{2}{\sqrt{6}}$ & $z = -\frac{1}{\pm\sqrt{6}} = \mp\frac{1}{\sqrt{6}}$

By ① $y = z - x = \mp\frac{1}{\sqrt{6}} - \left(\pm\frac{2}{\sqrt{6}}\right) = \mp\frac{3}{\sqrt{6}}$

so solutions to ①–⑤ are:

$\left(\frac{2}{\sqrt{6}}, -\frac{3}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$ or $\left(-\frac{2}{\sqrt{6}}, +\frac{3}{\sqrt{6}}, +\frac{1}{\sqrt{6}}\right)$

substitute into $f(x,y,z)$

$f\left(\frac{2}{\sqrt{6}}, \frac{-3}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right) = \frac{12}{\sqrt{6}}$ max

$f\left(-\frac{2}{\sqrt{6}}, +\frac{3}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = -\frac{12}{\sqrt{6}}$ min