CHAPTER TWO

Extreme Values of $\zeta(s)$

9. Introduction

We pointed out in §1 that the function S(t), being defined as $\pi^{-1}\mathrm{Im}\log\zeta(\frac{1}{2}+it)$, determines the finer structure of the vertical distribution of the zeros of $\zeta(s)$. However, in most of the applications, its significance is overshadowed by the horizontal distribution of the zeros (the zero density theorems). Under the RH, one can draw delicate conclusions about the distribution of primes if we know more about the distribution of the zeros on the critical line. For example, under the RH and the truth of the so-called "pair correlation" conjecture*, Heath-Brown and Goldston [5] recently proved that, the gaps between consecutive primes, $p_{n+1} - p_n$, is $o\{p_n^{\frac{1}{2}}(\log p_n)^{\frac{1}{2}}\}$.

The study of S(t) and the related function $S_1(t)$ (being defined as $\int_0^t S(u)du$) has begun earlier this century. For instance, Backlund [1] proved the classical result

$$S(t) = O(\log t)$$

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* This conjecture was proposed by Montgomery [10] in 1972.

in 1914 and Littlewood [8, 9] proved, among other things, that

$$\int_0^t S(u)du = S_1(t) = O(\log t) \tag{9.1}$$

and

$$\int_0^t |S(u)|du = O(t\log\log t).$$

These results of Littlewood suggested that S(t) is highly oscillatory and changes sign at high frequency.(Our Theorem 7.1 confirmed this.)

Among the various properties of S(t), the study of its order of infinity has attracted the most attention. From the familiar formula

$$N(t) = \vartheta(t) + S(t),$$

the order of S(t) tells how well the counting function N(t) is being approximated by $\vartheta(t)$. An exceptionally large value attained by S(t) corresponds to a cluster of zeros of $\zeta(s)$ over a short interval, while large and negative value of S(t) indicates the deficiency of zeros.

The study of S(t) has been very difficult because $\log\zeta(s)$ is a very complicated function. The most significant breakthrough after Littlewood was made by Selberg in the forties. In his papers [13, 14], he found useful formulas for S(t) (for example, Theorem 2 in [14]), from which he made remarkable improvements over some previous results. Our knowledge about the order of $S(t)$ is consisting of * :

$$S(t) = \begin{cases} O(\log t), \\ o(\log t) & \text{under the Lindelöf Hypothesis,} \\ O(\log t/\log\log t) & \text{under the RH} \end{cases}$$

and

$$S(t) = \begin{cases} \Omega_{\pm}\{(\log t)^{1/3}(\log\log t)^{-7/3}\} \ ^{**}, \\ \Omega_{\pm}\{(\log t/\log\log t)^{\frac{1}{2}}\} & \text{under the RH.} \end{cases}$$

For $S_1(t)$, we knew that

$$S_1(t) = \begin{cases} O(\log t), \\ o(\log t) & \text{under the Lindelöf Hypothesis,} \\ O\{(\log t)(\log\log t)^{-2}\} & \text{under the RH} \end{cases}$$

and

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* See chapters 9, 13 and 14 of [16] for a more detailed account.

** [14, Theorem 9].

$$S_1(t) = \begin{cases} \Omega_+\{(\log t)^{\frac{1}{2}}(\log\log t)^{-4}\} \; * , \\ \Omega_-\{(\log t)^{1/3}(\log\log t)^{-10/3}\} \; * , \\ \Omega_\pm\{(\log t)^{\frac{1}{2}}(\log\log t)^{-3/2}\} \text{ under the RH.} \end{cases}$$

The above O-results have been the record ever since they were first proved earlier this century. Those Ω-results were obtained by Selberg [13, 14] in the nineteen forties. By looking at the proofs of these results, we believe that, those Ω-results proved under the RH are possibly quite near the true maximum order of $S(t)$ and $S_1(t)$. So, under the RH, it is the O-results that leave much room for improvements.

In this chapter, we shall improve by a power of loglogt those unconditional Ω-results cited above. These improvements are achieved by introducing new techniques into the basic method of Selberg in [14]. We shall describe this in the next section.

We also pursue further study of the "roughness" of $S(t)$ and $S_1(t)$ by looking at the functions

$$S(t+h) - S(t) \quad \text{and} \quad S_1(t+h) - S_1(t),$$

where $t \in [T,2T]$ and $0 < h < 1/\log\log T$. We obtain Ω-results for these functions both under the RH and unconditionally.

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* [14, Theorems 10 & 11].

More generally, one may consider the functions $\mathrm{Re}\log\zeta(\sigma+it)$ and $\mathrm{Im}\log\zeta(\sigma+it)$ for $\sigma \geq \frac{1}{2}$. For $\mathrm{Re}\log\zeta(\sigma+it)$, the results of Montgomery [11] and Levinson [7] are already very sharp (see the table below). For $\mathrm{Im}\log\zeta(\sigma+it)$ with fixed $\sigma > \frac{1}{2}$, Montgomery has the best result. We shall consider the situation when $0 \leq \sigma - \frac{1}{2} \leq 1/\log\log t$.

For convenience of reference, we put together the best Ω-results we know to date in the following table.

Assume $T \to \infty$ and $0 < h \le 1/\log\log T$.

	under RH	unconditionally
(1) $S(t)$	$= \Omega_{\pm}\{(\log t / \log\log t)^{\frac{1}{2}}\}$	$\Omega_{\pm}\{(\log t / \log\log t)^{\frac{1}{3}}\}$
(2) $\sup\limits_{t \in [T,2T]} \pm\{S(t+h)-S(t)\} \ge$	$c(h\log T)^{\frac{1}{2}}$	$c(h\log T)^{\frac{1}{3}}$
(3) $\sup\limits_{t \in [T,2T]} \pm\{\mathrm{Im}\log\zeta(\sigma+it)\} \ge$	$c(\log T)^{1-\sigma}(\log\log T)^{-\sigma}$	$c(\log T / \log\log T)^{\frac{1}{3}}$ for $0 \le \sigma-\frac{1}{2} \le (\log\log T / \log T)^{\frac{1}{3}}$, $c\{(\sigma-\frac{1}{2})\log T / \log\log T\}^{\frac{1}{2}}$ for $(\log\log T/\log T)^{\frac{1}{3}} < \sigma-\frac{1}{2} \le (\log\log T)^{-1}$
(4) $\sup\limits_{t \in [T,2T]}\{\log\lvert\zeta(\sigma+it)\rvert\} \ge$	$c(\log T)^{1-\sigma}(\log\log T)^{-\sigma}$	$c(\log T)^{1-\sigma}/\log\log T$
(5) $S_1(t)$	$= \Omega_{\pm}\{(\log t)^{\frac{1}{2}}(\log\log t)^{-\frac{3}{2}}\}$	$\Omega_{+}\{(\log t)^{\frac{1}{2}}(\log\log t)^{-\frac{9}{4}}\}$ $\Omega_{-}\{(\log t)^{\frac{1}{3}}(\log\log t)^{-\frac{4}{3}}\}$
(6) $\sup\limits_{t \in [T,2T]} \pm\{S_1(t+h)-S_1(t)\} \ge$	$ch(\log T / \log\log T)^{\frac{1}{2}}$	$ch(\log T / \log\log T)^{\frac{1}{3}}$

Remarks.

(a) All the unconditional results, except (4), are new results proved in this thesis.

(b) The conditional results in (3) and (4) were proved by Montgomery in [11]. He also proved some Ω-results for fixed $\sigma > \frac{1}{2}$.

(c) The unconditional result in (4) was proved by Levinson [7].

(d) (1) is an easy consequence of (6), see §14.

(e) The result that $S_1(t) = \Omega_+\{(\log t)^{\frac{1}{2}}(\log\log t)^{-9/4}\}$ is quite interesting, because it is already very close to what we can obtain under the RH.

In the next section, we shall give a brief sketch of the underlying method of Selberg published in [14] and those new techniques that bring us the improvements. In §11, we prove some preliminary lemmas and then in §§12-14, we shall give the details of the proofs of the varies Ω-theorems.

10. The Method of Proving Ω-theorems

The major difficulty one encountered in the study of S(t) is the lacking of convenient explicit formula to express it. The formula in Theorem 2 of [14] is quite sufficient for some problems, but it is useless for proving Ω-theorems because the remainder term there is too big.

Our method starts by taking the convolution of $\log\zeta(\frac{1}{2}+iu)$ with a kernel V(u) over the critical line. By means of the familiar residue theorem, we shift the path of integration towards the right to the line, say $\sigma = 2$, where $\log\zeta(s)$ is an absolutely convergent Dirichlet series. After interchanging the integration and summation, this integral becomes a Dirichlet series (or even a Dirichlet polynomial) which is our main term. In the process of shifting the path, those zeros of $\zeta(s)$ off the critical line, if exist, bring in a sum which we regard as a remainder. By taking the real parts (take imaginary parts if want to study $S_1(t)$ or $\log|\zeta(\sigma+it)|$), we have an equation of the form (see Lemma 11.1):

$$\int_{-\infty}^{\infty} S(t+u)V(u)du = W(t) + R(t)$$

$$+ \text{(a negligible term from the pole of } \zeta(s) \text{ at } s = 1).$$

Here,

$$W(t) = \mathrm{Im} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} V\left(-\frac{\log n}{2\pi}\right) n^{-\frac{1}{2}-it}$$

is the main term and R(t) is the remainder.

We then choose V(u) to be a positive function with a sharp peak at $u = 0$, for example, $V(u) = u^{-2}\sin^2(\tau u)$, τ large. The above integral is then "approximately" equals to S(t).

The next step is to choose t from [T,2T] suitably so that W(t) (or -W(t)) is big while $|R(t)|$ is small. This is a delicate procedure, because R(t) is a sum involving the zeros of $\zeta(s)$ and we only have very limited knowledge about their distribution.

In [14], Selberg derived Ω-theorems by the following argument. He showed that the integral

$$\int_T^{2T} \{W(t) - \psi\}^{2k} dt \ , \ \psi \text{ large},$$

is small so that the set of all those t in [T,2T] for which $W(t) \simeq \psi$ is big. On the other hand, by means of a zero density theorem, he showed that $|R(t)|$ is small, say $< \frac{1}{2}\psi$, except when t belongs to a small subset of [T,2T]. He then optimized the parameters k, ψ,...etc. so that this exceptional set has measure less than the set for which $W(t) \simeq \psi$. In this way, he obtained the desired Ω-results.

To achieve improvements, we use another argument. By Lemma 3.3 and a combinatorial argument, we derive a lower bound for the integral

$$\int_T^{2T} \{W(t)\}^{2k} dt.$$

Next, we consider the integral

$$\int_T^{2T} \{R(t)\}^{2k} dt \tag{10.1}$$

and use a zero density theorem (the same one Selberg used) to bound this from above. The key to derive Ω-results from these two estimates is the following observation (see Lemma 11.3). *If*

$$\int_T^{2T} \{W(t)\}^{2k} dt > TM_1^{2k}, \qquad \int_T^{2T} \{R(t)\}^{2k} dt \le TM_2^{2k}$$

and $M_1 \ge 2M_2$, *then there exists* $t \in [T,2T]$ *such that*

$$|W(t)| - |R(t)| > M_1 - M_2 \ge \tfrac{1}{2}M_1.$$

We adjust our parameters to maximize M_1 and, at the same time, keep M_2 smaller than $\frac{1}{2}M_1$. In this way we obtain the desired Ω-theorems.

We shall see in the proofs that the most delicate argument occurs in the upper estimation of the integral (10.1). However, if we assume the **RH**, R(t) does not exist. In that case, we simply maximize M_1 to deduce those conditional Ω-theorems we mentioned in the last section.

When the above procedure is applied to $S_1(t)$ we deduced that

$$S_1(t) = \Omega_+\{(\log t)^{\frac{1}{2}}(\log\log t)^{-5/2}\}.$$

To obtain the more precise result

$$S_1(t) = \Omega_+\{(\log)^{\frac{1}{2}}(\log\log t)^{-9/4}\},$$

we have to choose a new V(u) different from the one in [14] and introduce some other delicate arguments. The details of this will be discussed in §14.

11. Some Preliminary Lemmas

In this section, we shall prove some lemmas which will be used in the later sections.

Lemma 11.1. *Let* $\frac{1}{2} \le \sigma \le 2$. *Suppose* $V(x+iy)$ *is an analytic function in the horizontal strip :* $\sigma - 2 \le y \le 0$ *satisfying the growth condition*

$$\sup_{\sigma-2\le y\le 0} |V(x+iy)| = O\{(|x|\log^2|x|)^{-1}\}. \qquad (11.1)$$

For any $t \neq 0$, *we have*

$$\int_{-\infty}^{\infty} \log\zeta(\sigma+i(t+u))V(u)du = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} V(-\frac{\log n}{2\pi})n^{-\sigma-it}$$

$$+ 2\pi \sum_{\beta>\sigma} \int_0^{\beta-\sigma} V(\gamma-t-i\alpha)d\alpha + O(|t|^{-1}). \qquad (11.2)$$

Proof. Let Z be a large number. Consider the rectangle with vertices at $\sigma \pm iZ$, $2 \pm iZ$. From this rectangle, we remove the stretch joining σ and 1 and all those horizontal stretches joining $\sigma + i\gamma$ to $\beta + i\gamma$ for any zeros $\rho = \beta + i\gamma$ lying inside the rectangle. Call this domain R. Clearly, the function $\log\zeta(z)V(-t+i\sigma-iz)$ is single-valued and analytic in

R. Its values at the upper and lower side of a cut differ by $2\pi i V(\gamma - t - i(\alpha-\sigma))$, $\sigma \le \alpha \le \beta$. Integration along the boundary of R yields

$$\int_{-Z-t}^{Z-t} \log\zeta(\sigma+i(t+u))V(u)du = \int_{-Z-t}^{Z-t} \log\zeta(2+i(t+u))V(i(\sigma-2)+u)du$$

$$+ 2\pi \sum_{\substack{\sigma<\beta \\ -Z<\gamma\le Z}} \int_{\sigma}^{\beta} V(\gamma-t-i(\alpha-\sigma))d\alpha - 2\pi \int_{\min(1,\sigma)}^{1} V(-t-i(\alpha-\sigma))d\alpha$$

$$+ i\int_{\sigma}^{2} \{\log\zeta(\alpha+iZ)V(Z-t-i(\alpha-\sigma)) - \log\zeta(\alpha-iZ)V(-Z-t-i(\alpha-\sigma))\}d\alpha. \quad (11.3)$$

The last integral comes from integration along the top and the base of R. By (6.36) and the growth condition (11.1), we see easily that this integral tends to zero as $Z \to \infty$. In view of the fact that

$$\log\zeta(2+i(t+u)) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-2-i(t+u)},$$

we deduce from equation (11.3) that

$$\int_{-\infty}^{\infty} \log\zeta(\sigma+i(t+u))V(u)du = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \int_{-\infty}^{\infty} n^{-2-i(t+u)}V(i(\sigma-2)+u)du$$

$$+ 2\pi \sum_{\sigma<\beta} \int_{0}^{\beta-\sigma} V(\gamma-t-i\alpha)d\alpha - 2\pi \int_{\min(1-\sigma,0)}^{1-\sigma} V(-t-i\alpha)d\alpha. \quad (11.4)$$

The interchanging of integration and summation is justified by absolute convergence. By (11.1), the last term is $O(|t|^{-1})$, and we can shift the path of integration so that

$$\int_{-\infty}^{\infty} n^{-2-i(t+u)}V(i(\sigma-2)+u)du = \int_{-\infty}^{\infty} n^{-\sigma-i(t+u)}V(u)du = n^{-\sigma-it}\hat{V}(-\tfrac{\log n}{2\pi}).$$

Substitute this into (11.4), our lemma follows immediately. ∎

Lemma 11.2.* *Let* k *be a large positive integer,* $\lambda = 2k\log k$ *and* $5\lambda \le x^{3/4}$. *Suppose* $\{a_p\}_{p\le x}$ *is a sequence of numbers such that, there exists an absolute constant* $A \ge 1$ *satisfying*

$$A^{-1} \le |a_p/a_q| \le A \tag{11.5}$$

for all $p, q \in [\lambda, x^{3/4}]$, $q < p \le 2q$.

Then both

$$\int_T^{2T} \{\mathrm{Im} \sum_{p\le x} a_p p^{-it}\}^{2k} dt \quad \textit{and} \quad \int_T^{2T} \{\mathrm{Re} \sum_{p\le x} a_p p^{-it}\}^{2k} dt$$

$$\ge T(eA)^{-2k} k! (\sum_{\lambda<p\le x^{3/4}} |a_p|^2)^k - O\{(ck)^k (\sum_{p\le x} p|a_p|^2)^k\}. \tag{11.6}$$

Proof. It is sufficient to consider the first integral. We use the techniques developed in §3. For simplicity, write $\Sigma = \sum_{p\le x} a_p p^{-it}$. Apply binomial expansion to the expression

$$(\mathrm{Im}\Sigma)^{2k} = (2i)^{-2k}(\Sigma - \bar{\Sigma})^{2k},$$

we reduce the integral to a sum of integrals of the form

$$\int_T^{2T} \Sigma^{2k-\nu}\, \bar{\Sigma}^{\nu} dt \quad \text{for } \nu = 0, 1, \ldots, 2k.$$

By (3.8), this integral will have no main term if $\nu \ne k$. Hence, by (3.7)

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* The idea of this lemma is due to Selberg.

and (3.9), we have

$$\int_{T}^{2T} (\mathrm{Im}\Sigma)^{2k} dt = T2^{-2k}\binom{2k}{k} \sum_{\underline{p}\le x} |a_{\underline{p}}|^2 P(\underline{p}) + O\{\sum_{\nu=0}^{2k}\binom{2k}{\nu}\{\nu!(2k-\nu)!\}^{\frac{1}{2}}(\sum_{p\le x} p|a_p|^2)^k\}. \quad (11.7)$$

The remainder term is easily seen to be the one stated in (11.6). Therefore it remains to prove the desired lower bound for the main term. In particular, we look at the sum

$$\Psi = \sum_{\underline{p}\le x} |a_{\underline{p}}|^2 P(\underline{p}), \quad (11.8)$$

where $\underline{p}$ denotes $(p_1,\ldots,p_k)$, $|a_{\underline{p}}| = |a_{p_1}\cdots a_{p_k}|$ and $P(\underline{p})$ is the number of permutations of $p_1,\ldots,p_k$.

Let us divide the interval $(\lambda, x^{3/4}]$ into the following sequence of abutting sub-intervals:

$$\Delta_j := (2^{j-1}\lambda, 2^j\lambda] \quad \text{for } j = 1, 2,\ldots, m,$$

where m satisfies

$$\tfrac{1}{2}x^{3/4} < 2^m\lambda \le x^{3/4}.$$

Let $\underline{j} = (j_1,\ldots,j_k)$ and $\Psi_{\underline{j}} = \sum_{\underline{p}\in\Delta_{\underline{j}}} |a_{\underline{p}}|^2 P(\underline{p})$, where $\underline{p} \in \Delta_{\underline{j}}$ means $p_\nu \in \Delta_{j_\nu}$ for $\nu = 1, 2,\ldots, k$. Clearly

$$\Psi \ge \Sigma_{\underline{j}}\Psi_{\underline{j}},$$

where $\underline{j}$ runs over all k-tuples with integral entries between 1 and m.

Consider a fixed $\Psi_{\underline{j}}$. The assumption (11.5) guarantees that for any $\underline{p}$, $\underline{p}' \varepsilon \Delta_{\underline{j}}$, $|a_{\underline{p}}|$ and $|a_{\underline{p}'}|$ differ at most by a constant factor of the form B^k, with $A^{-1} \leq B \leq A$. So

$$\Psi_{\underline{j}} \geq |a_{\underline{q}}|^2 \sum_{\underline{p}\varepsilon\Delta_{\underline{j}}} P(\underline{p}),$$

where $|a_{\underline{q}}|^2$ is the smallest one among all the $|a_{\underline{p}}|^2$, $\underline{p} \varepsilon \Delta_{\underline{j}}$. Since $P(\underline{p}) \leq$ k! and equality holds when $\underline{p}$ has distinct entries, we have

$$\sum_{\underline{p}\varepsilon\Delta_{\underline{j}}} P(\underline{p}) \geq k!(\sum 1)\times(\text{chance of choosing a } \underline{p} \varepsilon \Delta_{\underline{j}} \text{ with all entries distinct}).$$

The chance of choosing a $\underline{p} \varepsilon \Delta_{\underline{j}}$ with all entries distinct is higher when Δ_{j_ν} ($\nu = 1, 2, \ldots, k$) contains more primes. Since Δ_1 is the shortest subinterval, it contains the least number of primes. Indeed, if the number of primes in Δ_1 is ℓ, then

$$\ell = \lambda/\log\lambda + O(\lambda/\log^2\lambda) \geq k,$$

by (0.1). The chance of choosing k distinct primes from Δ_1 is easily seen to be

$$\ell(\ell-1)\ldots(\ell-k+1)/\ell^k \geq \exp(-k^2/\ell) \geq e^{-k}.$$

Hence for any $\underline{j}$,

$$\sum_{\underline{p}\varepsilon\Delta_{\underline{j}}} P(\underline{p}) \geq k!e^{-k}(\sum_{\underline{p}\varepsilon\Delta_{\underline{j}}} 1)$$

and so, by (11.5),

$$\Psi_{\underline{j}} \geq k!e^{-k}|a_{\underline{q}}|^2 \sum_{p\in\Delta_{\underline{j}}} 1 \geq k!e^{-k}A^{-2k} \sum_{p\in\Delta_{\underline{j}}} |a_{\underline{p}}|^2.$$

Summing over all j, we have

$$\Psi \geq k!(eA^2)^{-k}\left(\sum_{\lambda<p\leq x^{1/4}} |a_p|^2\right)^k.$$

In view of (11.8), our lemma follows from this and (11.7). ∎

Lemma 11.3. *Let* k *be a positive integer and* $M_1 \geq 2M_2$. *Suppose* W(t) *and* R(t) *are real valued functions satisfying the following conditions :*

(i) $\int_T^{2T} \{W(t)\}^{2k} dt = TM_1^{2k}$,

(ii) $\left|\int_T^{2T} \{W(t)\}^{2k+1} dt\right| \leq \frac{1}{2}TM_1^{2k+1}$,

(iii) $\int_T^{2T} |R(t)|^{2k+1} dt = TM_2^{2k+1}$.

We have

$$\sup_{t\in[T,2T]} \{W(t) - |R(t)|\} \geq \tfrac{1}{2}M_1 - M_2$$

and

$$\inf_{t\in[T,2T]} \{W(t) + |R(t)|\} \leq -(\tfrac{1}{2}M_1 - M_2).$$

Proof. Let

$$W_+(t) = \max\{W(t), 0\} \quad \text{and} \quad W_-(t) = \min\{W(t), 0\}.$$

Conditions (i) and (ii) imply that $W(t)$ has to be large in both directions. Indeed, by (i) and Cauchy's inequality, we have

$$\int_T^{2T} |W(t)|^{2k+1} dt \geq TM_1^{2k+1}. \tag{11.9}$$

From the definitions of $W_+(t)$ and $W_-(t)$,

$$\int_T^{2T} |W_+(t)|^{2k+1} dt = \tfrac{1}{2}\int_T^{2T} |W(t)|^{2k+1} dt + \tfrac{1}{2}\int_T^{2T} \{W(t)\}^{2k+1} dt$$

and

$$\int_T^{2T} |W_-(t)|^{2k+1} dt = \tfrac{1}{2}\int_T^{2T} |W(t)|^{2k+1} dt - \tfrac{1}{2}\int_T^{2T} \{W(t)\}^{2k+1} dt.$$

Hence, in view of (ii) and (11.9), both

$$\int_T^{2T} |W_+(t)|^{2k+1} dt \quad \text{and} \quad \int_T^{2T} |W_-(t)|^{2k+1} dt \quad \geq \quad \tfrac{1}{4}TM_1^{2k+1}.$$

Let

$$\int_T^{2T} |W_+(t)|^{2k+1} dt = TM_3^{2k+1} \tag{11.10}$$

so that $M_3 > \frac{1}{2}M_1$. Write $m = 2k + 1$. By Cauchy's inequality, (11.10) and (iii), for any integer ν, $1 \leq \nu \leq m$, we have

$$\int_T^{2T} |W_+(t)|^{m-\nu} |R(t)|^{\nu-1} dt \leq TM_3^{m-\nu} M_2^{\nu-1}.$$

Hence

$$T(M_3^m - M_2^m) = \int_T^{2T} (|W_+(t)|^m - |R(t)|^m) dt$$

$$= \int_T^{2T} (|W_+(t)| - |R(t)|)(\sum_{\nu=1}^{m} |W_+(t)|^{m-\nu}|R(t)|^{\nu-1})dt$$

$$\leq \sup_{t\in[T,2T]} (|W_+(t)| - |R(t)|)\times(\sum_{\nu=1}^{m} TM_3^{m-\nu}M_2^{\nu-1})$$

$$= \sup_{t\in[T,2T]} \{W_+(t) - |R(t)|\}\times T(M_3^m - M_2^m)(M_3 - M_2)^{-1}.$$

That is

$$\sup_{t\in[T,2T]} \{W(t) - |R(t)|\} \geq M_3 - M_2 > \tfrac{1}{2}M_1 - M_2.$$

Similarly, we prove that

$$\inf_{t\in[T,2T]} \{W(t) + |R(t)|\} \leq -(\tfrac{1}{2}M_1 - M_2). \blacksquare$$

12. Ω-theorems for $\text{Im}\log\zeta(\sigma+it)$

Throughout this section, we assume $\sigma \geq \frac{1}{2}$ and $\sigma - \frac{1}{2} \to 0$ as $T \to \infty$. Our main result is

THEOREM 12.1.

$$\sup_{t\in[T,2T]} \pm\{\text{Im}\log\zeta(\sigma+it)\} \geq$$

$$\begin{cases} c(\log T/\log\log T)^{1/3} & \text{for } 0 \leq \sigma-\frac{1}{2} \leq (\log\log T/\log T)^{1/3}, \\ c\{(\sigma-\frac{1}{2})\log T/\log\log T\}^{\frac{1}{2}} & \text{for } (\log\log T/\log T)^{1/3} < \sigma-\frac{1}{2} \leq (\log\log T)^{-1}. \end{cases}$$

Corollary.

$$S(t) = \Omega_{\pm}\{(\log t/\log\log t)^{1/3}\}.$$

Remark. Montgomery [11, Cor. of Thm.1] has proved that : *for fixed* $\sigma > \frac{1}{2}$,

$$\text{Im}\log\zeta(\sigma+it) = \Omega_{\pm}\{(\log t)^{1-\sigma}(\log\log t)^{-\sigma}\}.$$

Our theorem gives information about $\operatorname{Im}\log\zeta(\sigma+it)$ in the vicinity of the critical line.

THEOREM 12.2.* *Under* RH,

$$S(t) = \Omega_{\pm}\{(\log t/\log\log t)^{\frac{1}{2}}\}.$$

We start with the proof of Theorem 12.1, as we shall see, Theorem 12.2 comes out easily at an early stage of the proof.

Let

$$T \to \infty, \quad \tau = 2\log\log T \quad \text{and} \quad t \in [T,2T]. \tag{12.1}$$

In Lemma 11.1, we choose

$$V(z) = (\tfrac{1}{2}z)^{-2}\sin^2(\tfrac{1}{2}\tau z).$$

By a standard integration, we show that its Fourier transform

$$\hat{V}(v) = 2\pi \max(0, \tau - |2\pi v|).$$

The imaginary part of equation (11.2) yields

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* This is Theorem 2 of [11].

$$\int_{-\infty}^{\infty}\{\mathrm{Im}\log\zeta(\sigma+i(t+u))\}(\tfrac{1}{2}u)^{-2}\sin^2(\tfrac{1}{2}\tau u)\,du = 2\pi\,\mathrm{Im}\sum_{n\le e^\tau}\frac{\Lambda(n)}{\log n}(\tau-\log n)n^{-\sigma-it}$$

$$+ 2\pi\sum_{\beta>\sigma}\int_0^{\beta-\sigma}\mathrm{Im}\{\frac{\sin(\frac{1}{2}\tau(\gamma-t-i\alpha))}{\frac{1}{2}\tau(\gamma-t-i\alpha)}\}^2 d\alpha + O(T^{-1}).$$

After a transformation of the variable of integration, it becomes

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\{\mathrm{Im}\log\zeta(\sigma+i(t+\frac{2u}{\tau}))\}u^{-2}\sin^2 u\,du = \mathrm{Im}\sum_{n\le e^\tau}\frac{\Lambda(n)}{\log n}(1-\frac{\log n}{\tau})n^{-\sigma-it}$$

$$+ \tau\sum_{\beta>\sigma}\int_0^{\beta-\sigma}\mathrm{Im}\{\frac{\sin(\frac{1}{2}\tau(\gamma-t-i\alpha))}{\frac{1}{2}\tau(\gamma-t-i\alpha)}\}^2 d\alpha + O(T^{-1}). \tag{12.2}$$

By (0.2) and (0.3),

$$\sum_{\substack{p^r\le e^\tau\\ r\ge 2}} p^{-r\sigma} = \sum_{p\le e^{\tau/2}} p^{-2\sigma} + O(1) = O(\log\tau).$$

Therefore the first sum on the right of (12.2) is

$$\mathrm{Im}\sum_{p\le e^\tau}(1-\frac{\log p}{\tau})p^{-\sigma-it} + O(\log\tau). \tag{12.3}$$

We already know that (see (8.4))

$$\mathrm{Im}\log\zeta(\sigma+it) = O(\log|t|) \quad \text{as } |t|\to\infty.$$

Consequently,

$$\int_{|u|>\log T}\{\mathrm{Im}\log\zeta(\sigma+i(t+\frac{2u}{\tau}))\}u^{-2}\sin^2 u\,du = O\{\int_{|u|>\log T}(\log T+\log u)u^{-2}du\} = O(1). \tag{12.4}$$

Since

$$\sup_{t\in[\frac{1}{2}T,3T]} \{\mathrm{Im}\log\zeta(\sigma+it)\} \geq \int_{|u|\leq \log T} \{\mathrm{Im}\log\zeta(\sigma+i(t+\tfrac{2u}{\tau}))\}u^{-2}\sin^2 u \, du, \qquad (12.5)$$

we deduce from (12.2), (12.3) and (12.4) that

$$\sup_{t\in[\frac{1}{2}T,3T]} \{\mathrm{Im}\log\zeta(\sigma+it)\} \geq \pi\{W(t) + R(t)\} - O(\log\tau) \quad \text{for all } t \in [T,2T], \qquad (12.6)$$

where

$$W(t) = \mathrm{Im} \sum_{p\leq e^\tau} (1 - \frac{\log p}{\tau}) p^{-\sigma-it}$$

and

$$R(t) = \tau \sum_{\beta>\sigma} \int_0^{\beta-\sigma} \mathrm{Im}\{\frac{\sin(\frac{1}{2}\tau(\gamma-t-i\alpha))}{\frac{1}{2}\tau(\gamma-t-i\alpha)}\}^2 d\alpha .$$

Let k be a positive integer such that

$$10k\log k \leq e^{2\tau/3} \quad \text{and} \quad e^{k\tau} \leq T^{1/16}. \qquad (12.7)$$

With this k, our W(t) which is of the form $\mathrm{Im}\,\Sigma a_p p^{-it}$, satisfies the hypotheses of Lemma 11.2 with $A = 3^\sigma$, so

$$\int_T^{2T} \{W(t)\}^{2k} dt \geq T(ck)^k \{\sum_{2k\log k<p\leq e^{3\tau/4}} p^{-2\sigma}(1 - \frac{\log p}{\tau})^2\}^k - O\{(ck)^k (\sum_{p\leq e^\tau} p^{1-2\sigma})^k\}.$$

By our assumption that $0 \leq \sigma - \frac{1}{2} \leq (\log\log T)^{-1} = 2/\tau$, the sum in the main term is

$$\gg \sum_{e^{2\tau/3}<p\leq e^{3\tau/4}} p^{-2\sigma} \geq e^{-3} \int_{\frac{4}{3}\tau(\sigma-\frac{1}{2})}^{\frac{3}{2}\tau(\sigma-\frac{1}{2})} \frac{dw}{w} - O(\tau^{-1}) = e^{-3}\log\tfrac{9}{8} - O(\tau^{-1}) \geq c.$$

The remainder term is $\ll (ck)^k e^{k\tau} \leq T^{1/16}(ck)^k$. Hence

$$\int_T^{2T} \{W(t)\}^{2k} dt \geq T(ck)^k. \tag{12.8}$$

By a familiar argument, we know that the integral $\int_T^{2T} \{W(t)\}^{2k+1} dt$ has no main term, and

$$\int_T^{2T} \{W(t)\}^{2k+1} dt << (ck)^{k+\frac{1}{2}} \Big(\sum_{p\leq e^\tau} p^{1-2\sigma}\Big)^{k+\frac{1}{2}} << T^{1/8} (ck)^k. \tag{12.9}$$

At this point, we see that if the RH were true, then $R(t) = 0$. By choosing $k = [c_1 \log T / \log\log T]$ (c_1 small) and applying Lemma 11.3, we conclude that

$$\sup_{t\in[T,2T]} \{W(t) - |R(t)|\} \geq \sqrt{(ck)} \geq c_2 (\log T / \log\log T)^{\frac{1}{2}}$$

for $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + (\log\log T)^{-1}$.

In view of (12.6), this proves half of Theorem 12.2. The other half is similarly proved.

Without assuming the RH, we have to bound the integral $\int_T^{2T} |R(t)|^{2k+1} dt$ from above and then appeal to Lemma 11.3.

For $t \in [T,2T]$, we define

$$\theta_t(\sigma) := \max_{\substack{\sigma<\beta \\ |\gamma-t|\leq \log^2 T}} (\beta-\sigma)$$

and

$$\theta_t := \theta_t(\tfrac{1}{2}) = \max_{\substack{\frac{1}{2}<\beta \\ |\gamma-t|\leq \log^2 T}} (\beta-\tfrac{1}{2}) .^*$$

Clearly,

$$0 \le \theta_t - \theta_t(\sigma) \le \sigma - \tfrac{1}{2}. \tag{12.10}$$

It is easy to check that, for any x, y ε $\mathbb{R}$,

$$\mathrm{Im}\{\frac{\sin(x+iy)}{x+iy}\}^2 = O\{|y|e^{2|y|}(1+x^2+y^2)^{-1}\}.$$

Moreover, for fixed x, $e^y(1+x^2+y^2)^{-1}$ is an increasing function of y. Hence, starting from the definition of R(t), we have

$$R(t) = O\{\tau \sum_{\beta>\sigma} \int_0^{\beta-\sigma} \tau\alpha e^{\tau\alpha}\{1 + (\tfrac{1}{2}\tau)^2(\gamma-t)^2 + (\tfrac{1}{2}\tau)^2\alpha^2\}^{-1}d\alpha\}$$

$$= O\{\sum_{\beta>\sigma} (\beta-\sigma)^2 e^{\tau(\beta-\sigma)}\{(2/\tau)^2 + (\gamma-t)^2 + (\beta-\sigma)^2\}^{-1}\}$$

$$= O\{\theta_t^2(\sigma) \sum_{\substack{\sigma<\beta \\ |\gamma-t|\le \log^2 T}} e^{\tau(\beta-\sigma)}\{(2/\tau)^2 + (\gamma-t)^2 + (\beta-\sigma)^2\}^{-1}$$

$$+ e^{\frac{1}{2}\tau} \sum_{|\gamma-t|>\log^2 T} (\gamma-t)^{-2}\}$$

$$= O\{\theta_t^2(\sigma)e^{\tau\theta_t} \sum_{\substack{\frac{1}{2}<\beta \\ |\gamma-t|\le \log^2 T}} \{(2/\tau)^2 + (\gamma-t)^2 + \theta_t^2\}^{-1} + \log T \sum_{|\gamma-t|>\log^2 T} (\gamma-t)^{-2}\}. \tag{12.11}$$

In §7 of [14], Selberg has proved that

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* The same θ_t has been defined by Selberg in [14, §7].

$$\sum_{\substack{\frac{1}{2}<\beta \\ |\gamma-t|\le \log^2 T}} \{(2/\tau)^2 + (\gamma-t)^2 + \theta_t^2\}^{-1} = O(\tau\log T).$$

The other sum in (12.11) is easily seen to be $O(1/\log T)$, because $N(t+1) - N(t) \asymp c\log t$. Moreover, by (12.10) and (12.1), we have $0 \le \tau\theta_t - \tau\theta_t(\sigma) \le \tau(\sigma-\frac{1}{2}) \le 2$. All these estimates together implies that

$$R(t) = O\{\theta_t^2(\sigma)e^{\tau\theta_t(\sigma)}\tau\log T + 1\}.$$

By definition of $\theta_t(\sigma)$

$$\int_T^{2T}\{\theta_t^2(\sigma)e^{\tau\theta_t(\sigma)}\}^{2k+1}dt \le e^{\frac{1}{2}\tau}\log^2 T \sum_{\substack{\sigma<\beta \\ \frac{1}{2}T<\gamma\le 3T}} (\beta-\sigma)^{4k+2}e^{2k\tau(\beta-\sigma)}.$$

In view of (12.7), we can apply Lemma 5.1 to the last sum, the conclusion is

$$\sum_{\substack{\sigma<\beta \\ \frac{1}{2}T<\gamma\le 3T}} (\beta-\sigma)^{4k+2}e^{2k\tau(\beta-\sigma)} \ll T^{1-\frac{1}{4}(\sigma-\frac{1}{2})}(ck)^{4k+2}(\log T)^{-4k-1}.$$

Hence we have

$$\int_T^{2T}|R(t)|^{2k+1}dt \le T^{1-\frac{1}{4}(\sigma-\frac{1}{2})}(c\tau k^2/\log T)^{2k+1}\log^4 T.$$

By (12.6), (12.8), (12.9) and Lemma 11.3, we see that if

$$\sqrt{k} > (c_1\tau k^2/\log T)\{T^{-\frac{1}{4}(\sigma-\frac{1}{2})}\log^4 T\}^{1/(2k+1)} \qquad (12.12)$$

for some large constant c_1, then

$$\sup_{t\in[\frac{1}{2}T,3T]} \{\operatorname{Im}\log\zeta(\sigma+it)\} \geq c\sqrt{k}. \tag{12.13}$$

We consider two situations.

(i) $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + (\log\log T\ /\log T)^{1/3}$.

We take $k = [c_2(\log T\ /\log\log T)^{2/3}]$, where $c_2 = (4c_1)^{-2/3}$.

(ii) $\frac{1}{2} + (\log\log T\ /\log T)^{1/3} < \sigma \leq \frac{1}{2} + (\log\log T)^{-1}$.

We take $k = [(\sigma-\frac{1}{2})\log T\ /16\log\log T]$.

It is easy to check that in each case, the values of k and τ satisfy the conditions (12.7) and (12.12). Thus, the Ω_+ result of Theorem 12.1 follows from (12.13). The Ω_- result is proved by the same argument.

13. Ω-theorems for $S(t+h) - S(t)$

A natural way to demonstrate the "roughness" of a function is to show that it has big rises and big falls over short intervals. This suggests us to consider the Ω results of $S(t+h) - S(t)$ for small h. There is not much literature concerning this problem, although Selberg has proved that (unpublished): *under the* RH

$$\sup_{t\varepsilon[T,2T]} \pm\{S(t+h) - S(t)\} \geq c(h\log T)^{\frac{1}{2}} \tag{13.1}$$

for fixed $h \varepsilon [(\log T)^{-1}, (\log\log T)^{-1}]$.

In this section, we shall prove a corresponding theorem unconditionally. Our main result is

THEOREM 13.1. *For any large* T *and any fixed* h *in the interval* $[(\log T)^{-1}, (\log\log T)^{-1}]$, *we have*

$$\sup_{t\varepsilon[T,2T]} \pm\{S(t+h) - S(t)\} \geq c(h\log T)^{1/3}.$$

Before we prove this, we shall first prove another theorem which will be used in the proof of Theorem 13.1. The theorem itself has some independent interest.

THEOREM 13.2. *Suppose* $T \to \infty$, $T^{\eta} < H \le T$, $\eta > \frac{1}{2}$ *and* $0 < h < 1$. *For any positive integer* k, *we have*

$$\int_T^{T+H} \{S(t+h) - S(t)\}^{2k} dt = HA_k\{\log(2 + h\log T)\}^k$$

$$+ O\{H(ck)^k\{k^k + \{\log(2 + h\log T)\}^{k-\frac{1}{2}}\}\},$$

where $A_k = (2k)!/(2^k\pi^{2k}k!)$.

Corollary.

$$\int_T^{2T}\{S(t+h) - S(t)\}^{2k}dt = O\{T(ck)^k\{k^k + \{\log(2 + h\log T)\}^k\}\}.$$

We shall use this corollary in the proof of Theorem 13.1.

Proof of Theorem 13.2.

Let

$$0 < \varepsilon < (\eta - \tfrac{1}{2})/48,$$

$$x = T^{\varepsilon/k},$$

$$Q(t) = S(t) - \pi^{-1}\operatorname{Im}\sum_{p\le x} p^{-\frac{1}{2}-it} \qquad (13.2)$$

and

$$P(t) = \pi^{-1} \mathrm{Im} \sum_{p \le x} p^{-\frac{1}{2}-it}(p^{-ih} - 1). \tag{13.3}$$

Apply (5.1) with $\sigma = \frac{1}{2}$, we have

$$\int_T^{T+H} \{Q(t)\}^{2k} dt = O\{H(ck)^{2k}\}.$$

Let

$$U(t) = Q(t+h) - Q(t), \tag{13.4}$$

then

$$\int_T^{T+H} \{U(t)\}^{2k} dt = O\{H(ck)^{2k}\}. \tag{13.5}$$

We now use the familiar technique discussed in §3 to estimate the integral $\int_T^{T+H} \{P(t)\}^{2k} dt$. Write

$$P(t) = (2i)^{-1}\{(\sum_{p \le x} a_p p^{-it}) - \overline{(\sum_{p \le x} a_p p^{-it})}\},$$

where

$$a_p = \pi^{-1} p^{-\frac{1}{2}}(p^{-ih} - 1) \quad \text{for} \quad p \le x.$$

First apply binomial expansion and then integrate term by term, we conclude from Lemma 3.3 that

$$\int_T^{T+H} \{P(t)\}^{2k} dt = H\Psi + r, \tag{13.6}$$

where

$$\Psi = 2^{-2k}\binom{2k}{k}\sum_{\underline{p}\le x} |a_{\underline{p}}|^2 P(\underline{p}) \ *$$

and

$$r = O\{2^{-2k}\sum_{\nu=0}^{2k}\binom{2k}{\nu}\{(2k-\nu)!\nu!\}^{\frac{1}{2}}(\sum_{p\le x} p|a_p|^2)^k\} = O\{(ckx)^k\}.$$

The sum in Ψ is

$$k!(\sum_{\underline{p}\le x} |a_{\underline{p}}|^2) - O\{k!(\sum_{\substack{\underline{p}\le x \\ \text{not all entries} \\ \text{of } \underline{p} \text{ are distinct}}} |a_{\underline{p}}|^2)\}$$

$$= k!\sum_{\underline{p}\le x} |a_{\underline{p}}|^2 - O\{k!k^2(\sum_{p\le x} |a_p|^4)(\sum_{p\le x} |a_p|^2)^{k-2}\}$$

$$= k!(\sum_{p\le x} |a_p|^2)^k + O\{(ck)^k(\sum_{p\le x} |a_p|^2)^{k-2}\}.$$

Plainly,

$$|a_p|^2 = 2\pi^{-2}p^{-1}\{1 - \cos(h\log p)\} \quad \text{for } p \le x.$$

Using (0.2) and (0.7), we showed that

$$\sum_{p\le x} |a_p|^2 = 2\pi^{-2}\{\sum_{p\le x} p^{-1} - \sum_{p\le x} p^{-1}\cos(h\log p)\} = 2\pi^{-2}\log(2 + h\log T) + O(\log k).$$

----

* The notation is from Lemma 3.3, that is, $\underline{p}$ denotes $(p_1,\ldots,p_k)$, $a_{\underline{p}} = a_{p_1}\ldots a_{p_k}$ and $P(\underline{p})$ is the number of permutations of $p_1,\ldots,p_k$.

Hence

$$\Psi = A_k\{\log(2 + h\log T) + O(\log k)\}^k + O\{(ck)^k\{\log(2 + h\log T) + \log k\}^{k-2}\}$$

$$= A_k\{\log(2 + h\log T)\}^k + O\{(ck)^k\{(\log(2 + h\log T))^{k-1} + (\log k)^k\}\}.$$

Thus, from (13.6), we have

$$\int_T^{T+H}\{P(t)\}^{2k}dt = HA_k\{\log(2 + h\log T)\}^k$$
$$+ O\{H(ck)^k\{(\log(2 + h\log T))^{k-1} + (\log k)^k\}\} \qquad (13.7)$$

and

$$\int_T^{T+H}\{P(t)\}^{2k}dt = O\{H(ck)^k\{\log(2 + h\log T)\}^k\}. \qquad (13.8)$$

From the definitions (13.2), (13.3) and (13.4), we have $S(t+h) - S(t) = U(t) + P(t)$. Thus

$$\int_T^{T+H}\{S(t+h) - S(t)\}^{2k}dt = \int_T^{T+H}\{P(t)\}^{2k}dt$$
$$+ O\{c^k\int_T^{T+H}|P(t)|^{2k-1}|U(t)|dt + c^k\int_T^{T+H}\{U(t)\}^{2k}dt\}.$$

By (13.5), (13.8) and Cauchy's inequality, the first integral in the remainder is

$$\ll \{\int_T^{T+H}\{P(t)\}^{2k}dt\}^{(2k-1)/2k}\{\int_T^{T+H}\{U(t)\}^{2k}dt\}^{1/2k}$$

$$\ll H(ck)^k\{\log(2 + h\log T)\}^{k-\frac{1}{2}}.$$