

CHAPTER TWO

Extreme Values of $\zeta(s)$

9. Introduction

We pointed out in §1 that the function $S(t)$, being defined as $\pi^{-1} \text{Im} \log \zeta(\frac{1}{2} + it)$, determines the finer structure of the vertical distribution of the zeros of $\zeta(s)$. However, in most of the applications, its significance is overshadowed by the horizontal distribution of the zeros (the zero density theorems). Under the RH, one can draw delicate conclusions about the distribution of primes if we know more about the distribution of the zeros on the critical line. For example, under the RH and the truth of the so-called "pair correlation" conjecture*, Heath-Brown and Goldston [5] recently proved that, the gaps between consecutive primes, $p_{n+1} - p_n$, is $o\{p_n^{\frac{1}{2}}(\log p_n)^{\frac{1}{2}}\}$.

The study of $S(t)$ and the related function $S_1(t)$ (being defined as $\int_0^t S(u) du$) has begun earlier this century. For instance, Backlund [1] proved the classical result

$$S(t) = O(\log t)$$

* This conjecture was proposed by Montgomery [10] in 1972.

in 1914 and Littlewood [8, 9] proved, among other things, that

$$\int_0^t S(u)du = S_1(t) = O(\log t) \quad (9.1)$$

and

$$\int_0^t |S(u)|du = O(t \log \log t).$$

These results of Littlewood suggested that $S(t)$ is highly oscillatory and changes sign at high frequency. (Our Theorem 7.1 confirmed this.)

Among the various properties of $S(t)$, the study of its order of infinity has attracted the most attention. From the familiar formula

$$N(t) = \mathfrak{J}(t) + S(t),$$

the order of $S(t)$ tells how well the counting function $N(t)$ is being approximated by $\mathfrak{J}(t)$. An exceptionally large value attained by $S(t)$ corresponds to a cluster of zeros of $\zeta(s)$ over a short interval, while large and negative value of $S(t)$ indicates the deficiency of zeros.

The study of $S(t)$ has been very difficult because $\log \zeta(s)$ is a very complicated function. The most significant breakthrough after Littlewood was made by Selberg in the forties. In his papers [13, 14], he found useful formulas for $S(t)$ (for example, Theorem 2 in [14]), from which he made remarkable improvements over some previous results. Our knowledge

about the order of $S(t)$ is consisting of * :

$$S(t) = \begin{cases} O(\log t), \\ o(\log t) \text{ under the Lindelöf Hypothesis,} \\ O(\log t / \log \log t) \text{ under the RH} \end{cases}$$

and

$$S(t) = \begin{cases} \Omega_{\pm}\{(\log t)^{1/3}(\log \log t)^{-7/3}\}^{**}, \\ \Omega_{\pm}\{(\log t / \log \log t)^{\frac{1}{2}}\} \text{ under the RH.} \end{cases}$$

For $S_1(t)$, we knew that

$$S_1(t) = \begin{cases} O(\log t), \\ o(\log t) \text{ under the Lindelöf Hypothesis,} \\ O\{(\log t)(\log \log t)^{-2}\} \text{ under the RH} \end{cases}$$

and

* See chapters 9, 13 and 14 of [16] for a more detailed account.

** [14, Theorem 9].

$$S_1(t) = \begin{cases} \Omega_+ \{ (\log t)^{\frac{1}{2}} (\log \log t)^{-4} \} * , \\ \Omega_- \{ (\log t)^{1/3} (\log \log t)^{-10/3} \} * , \\ \Omega_{\pm} \{ (\log t)^{\frac{1}{2}} (\log \log t)^{-3/2} \} \text{ under the RH.} \end{cases}$$

The above 0-results have been the record ever since they were first proved earlier this century. Those Ω -results were obtained by Selberg [13, 14] in the nineteen forties. By looking at the proofs of these results, we believe that, those Ω -results proved under the RH are possibly quite near the true maximum order of $S(t)$ and $S_1(t)$. So, under the RH, it is the 0-results that leave much room for improvements.

In this chapter, we shall improve by a power of $\log \log t$ those unconditional Ω -results cited above. These improvements are achieved by introducing new techniques into the basic method of Selberg in [14]. We shall describe this in the next section.

We also pursue further study of the "roughness" of $S(t)$ and $S_1(t)$ by looking at the functions

$$S(t+h) - S(t) \text{ and } S_1(t+h) - S_1(t),$$

where $t \in [T, 2T]$ and $0 < h < 1/\log \log T$. We obtain Ω -results for these functions both under the RH and unconditionally.

* [14, Theorems 10 & 11].

More generally, one may consider the functions $\operatorname{Re} \log \zeta(\sigma+it)$ and $\operatorname{Im} \log \zeta(\sigma+it)$ for $\sigma \geq \frac{1}{2}$. For $\operatorname{Re} \log \zeta(\sigma+it)$, the results of Montgomery [11] and Levinson [7] are already very sharp (see the table below). For $\operatorname{Im} \log \zeta(\sigma+it)$ with fixed $\sigma > \frac{1}{2}$, Montgomery has the best result. We shall consider the situation when $0 \leq \sigma - \frac{1}{2} \leq 1/\log \log t$.

For convenience of reference, we put together the best Ω -results we know to date in the following table.

Assume $T \rightarrow \infty$ and $0 < h \leq 1/\log\log T$.

under RH

unconditionally

$$(1) S(t) = \Omega_{\pm}\{(\log t / \log\log t)^{\frac{1}{2}}\} \quad \Omega_{\pm}\{(\log t / \log\log t)^{\frac{1}{3}}\}$$

$$(2) \sup_{t \in [T, 2T]}^{\pm} \{S(t+h) - S(t)\} \geq c(h \log T)^{\frac{1}{2}} \quad c(h \log T)^{\frac{1}{3}}$$

$$(3) \sup_{t \in [T, 2T]}^{\pm} \{\operatorname{Im} \log \zeta(\sigma + it)\} \geq c(\log T)^{1-\sigma} (\log\log T)^{-\sigma} \cdot c(\log T / \log\log T)^{\frac{1}{3}} \text{ for } 0 \leq \sigma - \frac{1}{2} \leq (\log\log T / \log T)^{\frac{1}{3}},$$

$$c\{(\sigma - \frac{1}{2}) \log T / \log\log T\}^{\frac{1}{2}} \text{ for } (\log\log T / \log T)^{\frac{1}{3}} < \sigma - \frac{1}{2} \leq (\log\log T)^{-1}$$

$$(4) \sup_{t \in [T, 2T]} \{\log |\zeta(\sigma + it)|\} \geq c(\log T)^{1-\sigma} (\log\log T)^{-\sigma} \quad c(\log T)^{1-\sigma} / \log\log T$$

$$(5) S_1(t) = \Omega_{\pm}\{(\log t)^{\frac{1}{2}} (\log\log t)^{-\frac{3}{2}}\} \quad \Omega_{\pm}\{(\log t)^{\frac{1}{2}} (\log\log t)^{-\frac{3}{4}}\}$$

$$\Omega_{\pm}\{(\log t)^{\frac{1}{2}} (\log\log t)^{-\frac{4}{3}}\}$$

$$(6) \sup_{t \in [T, 2T]}^{\pm} \{S_1(t+h) - S_1(t)\} \geq ch(\log T / \log\log T)^{\frac{1}{2}} \quad ch(\log T / \log\log T)^{\frac{1}{3}}$$

Remarks.

- (a) All the unconditional results, except (4), are new results proved in this thesis.
- (b) The conditional results in (3) and (4) were proved by Montgomery in [11]. He also proved some Ω -results for fixed $\sigma > \frac{1}{2}$.
- (c) The unconditional result in (4) was proved by Levinson [7].
- (d) (1) is an easy consequence of (6), see §14.
- (e) The result that $S_1(t) = \Omega_+\{(\log t)^{\frac{1}{2}}(\log \log t)^{-9/4}\}$ is quite interesting, because it is already very close to what we can obtain under the RH.

In the next section, we shall give a brief sketch of the underlying method of Selberg published in [14] and those new techniques that bring us the improvements. In §11, we prove some preliminary lemmas and then in §§12-14, we shall give the details of the proofs of the various Ω -theorems.

10. The Method of Proving Ω -theorems

The major difficulty one encountered in the study of $S(t)$ is the lacking of convenient explicit formula to express it. The formula in Theorem 2 of [14] is quite sufficient for some problems, but it is useless for proving Ω -theorems because the remainder term there is too big.

Our method starts by taking the convolution of $\log\zeta(\frac{1}{2}+iu)$ with a kernel $V(u)$ over the critical line. By means of the familiar residue theorem, we shift the path of integration towards the right to the line, say $\sigma = 2$, where $\log\zeta(s)$ is an absolutely convergent Dirichlet series. After interchanging the integration and summation, this integral becomes a Dirichlet series (or even a Dirichlet polynomial) which is our main term. In the process of shifting the path, those zeros of $\zeta(s)$ off the critical line, if exist, bring in a sum which we regard as a remainder. By taking the real parts (take imaginary parts if want to study $S_1(t)$ or $\log|\zeta(\sigma+it)|$), we have an equation of the form (see Lemma 11.1):

$$\int_{-\infty}^{\infty} S(t+u)V(u)du = W(t) + R(t)$$

+ (a negligible term from the pole of $\zeta(s)$ at $s = 1$).

Here,

$$W(t) = \lim_{\infty} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} V\left(-\frac{\log n}{2\pi}\right) n^{-\frac{1}{2}-it}$$

is the main term and $R(t)$ is the remainder.

We then choose $V(u)$ to be a positive function with a sharp peak at $u = 0$, for example, $V(u) = u^{-2} \sin^2(\tau u)$, τ large. The above integral is then "approximately" equals to $S(t)$.

The next step is to choose t from $[T, 2T]$ suitably so that $W(t)$ (or $-W(t)$) is big while $|R(t)|$ is small. This is a delicate procedure, because $R(t)$ is a sum involving the zeros of $\zeta(s)$ and we only have very limited knowledge about their distribution.

In [14], Selberg derived Ω -theorems by the following argument. He showed that the integral

$$\int_T^{2T} \{W(t) - \psi\}^{2k} dt, \quad \psi \text{ large,}$$

is small so that the set of all those t in $[T, 2T]$ for which $W(t) \approx \psi$ is big. On the other hand, by means of a zero density theorem, he showed that $|R(t)|$ is small, say $< \frac{1}{2}\psi$, except when t belongs to a small subset of $[T, 2T]$. He then optimized the parameters k, ψ, \dots etc. so that this exceptional set has measure less than the set for which $W(t) \approx \psi$. In this way, he obtained the desired Ω -results.

To achieve improvements, we use another argument. By Lemma 3.3 and a combinatorial argument, we derive a lower bound for the integral

$$\int_T^{2T} \{W(t)\}^{2k} dt.$$

Next, we consider the integral

$$\int_T^{2T} \{R(t)\}^{2k} dt \tag{10.1}$$

and use a zero density theorem (the same one Selberg used) to bound this from above. The key to derive Ω -results from these two estimates is the following observation (see Lemma 11.3). *If*

$$\int_T^{2T} \{W(t)\}^{2k} dt > TM_1^{2k}, \quad \int_T^{2T} \{R(t)\}^{2k} dt \leq TM_2^{2k}$$

and $M_1 \geq 2M_2$, then there exists $t \in [T, 2T]$ such that

$$|W(t)| - |R(t)| > M_1 - M_2 \geq \frac{1}{2}M_1.$$

We adjust our parameters to maximize M_1 and, at the same time, keep M_2 smaller than $\frac{1}{2}M_1$. In this way we obtain the desired Ω -theorems.

We shall see in the proofs that the most delicate argument occurs in the upper estimation of the integral (10.1). However, if we assume the RH, $R(t)$ does not exist. In that case, we simply maximize M_1 to deduce those conditional Ω -theorems we mentioned in the last section.

When the above procedure is applied to $S_1(t)$ we deduced that

$$S_1(t) = \Omega_+ \{ (\log t)^{\frac{1}{2}} (\log \log t)^{-5/2} \}.$$

To obtain the more precise result

$$S_1(t) = \Omega_+ \{ (\log)^{\frac{1}{2}} (\log \log t)^{-9/4} \},$$

we have to choose a new $V(u)$ different from the one in [14] and introduce some other delicate arguments. The details of this will be discussed in §14.

11. Some Preliminary Lemmas

In this section, we shall prove some lemmas which will be used in the later sections.

Lemma 11.1. *Let $\frac{1}{2} \leq \sigma \leq 2$. Suppose $V(x+iy)$ is an analytic function in the horizontal strip : $\sigma - 2 \leq y \leq 0$ satisfying the growth condition*

$$\sup_{\sigma-2 \leq y \leq 0} |V(x+iy)| = O(|x| \log^2 |x|^{-1}). \quad (11.1)$$

For any $t \neq 0$, we have

$$\int_{-\infty}^{\infty} \log \zeta(\sigma+i(t+u))V(u)du = \sum_{n \neq 2}^{\infty} \frac{\Lambda(n)}{\log n} V\left(-\frac{\log n}{2\pi}\right)n^{-\sigma-it} \\ + 2\pi \sum_{\beta > \sigma} \int_0^{\beta-\sigma} V(\gamma-t-i\alpha)d\alpha + O(|t|^{-1}). \quad (11.2)$$

Proof. Let Z be a large number. Consider the rectangle with vertices at $\sigma \pm iZ$, $2 \pm iZ$. From this rectangle, we remove the stretch joining σ and 1 and all those horizontal stretches joining $\sigma + i\gamma$ to $\beta + i\gamma$ for any zeros $\rho = \beta + i\gamma$ lying inside the rectangle. Call this domain R . Clearly, the function $\log \zeta(z)V(-t+i\sigma-iz)$ is single-valued and analytic in

R. Its values at the upper and lower side of a cut differ by $2\pi iV(\gamma-t-i(\alpha-\sigma))$, $\sigma \leq \alpha \leq \beta$. Integration along the boundary of R yields

$$\begin{aligned} \int_{-Z-t}^{Z-t} \log\zeta(\sigma+i(t+u))V(u)du &= \int_{-Z-t}^{Z-t} \log\zeta(2+i(t+u))V(i(\sigma-2)+u)du \\ &+ 2\pi \sum_{\substack{\sigma < \beta \\ -Z < \gamma < Z}} \int_{\sigma}^{\beta} V(\gamma-t-i(\alpha-\sigma))d\alpha - 2\pi \int_{\min(1,\sigma)}^1 V(-t-i(\alpha-\sigma))d\alpha \\ &+ i \int_{\sigma}^2 \{ \log\zeta(\alpha+iZ)V(Z-t-i(\alpha-\sigma)) - \log\zeta(\alpha-iZ)V(-Z-t-i(\alpha-\sigma)) \} d\alpha. \end{aligned} \quad (11.3)$$

The last integral comes from integration along the top and the base of R. By (6.36) and the growth condition (11.1), we see easily that this integral tends to zero as $Z \rightarrow \infty$. In view of the fact that

$$\log\zeta(2+i(t+u)) = \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} n^{-2-i(t+u)},$$

we deduce from equation (11.3) that

$$\begin{aligned} \int_{-\infty}^{\infty} \log\zeta(\sigma+i(t+u))V(u)du &= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \int_{-\infty}^{\infty} n^{-2-i(t+u)}V(i(\sigma-2)+u)du \\ &+ 2\pi \sum_{\sigma < \beta} \int_0^{\beta-\sigma} V(\gamma-t-i\alpha)d\alpha - 2\pi \int_{\min(1-\sigma,0)}^{1-\sigma} V(-t-i\alpha)d\alpha. \end{aligned} \quad (11.4)$$

The interchanging of integration and summation is justified by absolute convergence. By (11.1), the last term is $O(|t|^{-1})$, and we can shift the path of integration so that

$$\int_{-\infty}^{\infty} n^{-2-i(t+u)}V(i(\sigma-2)+u)du = \int_{-\infty}^{\infty} n^{-\sigma-i(t+u)}V(u)du = n^{-\sigma-it} \hat{V}\left(-\frac{\log n}{2\pi}\right).$$

Substitute this into (11.4), our lemma follows immediately. ■

Lemma 11.2.* Let k be a large positive integer, $\lambda = 2k \log k$ and $5\lambda \leq x^{3/4}$. Suppose $\{a_p\}_{p \leq x}$ is a sequence of numbers such that, there exists an absolute constant $A \geq 1$ satisfying

$$A^{-1} \leq |a_p/a_q| \leq A \quad (11.5)$$

for all $p, q \in [\lambda, x^{3/4}]$, $q < p \leq 2q$.

Then both

$$\int_T^{2T} \{\operatorname{Im} \sum_{p \leq x} a_p p^{-it}\}^{2k} dt \quad \text{and} \quad \int_T^{2T} \{\operatorname{Re} \sum_{p \leq x} a_p p^{-it}\}^{2k} dt$$

$$\geq T(eA)^{-2k} k! \left(\sum_{\lambda < p \leq x} |a_p|^2 \right)^k - O\left\{ (ck)^k \left(\sum_{p \leq x} p |a_p|^2 \right)^k \right\}. \quad (11.6)$$

Proof. It is sufficient to consider the first integral. We use the techniques developed in §3. For simplicity, write $\Sigma = \sum_{p \leq x} a_p p^{-it}$. Apply binomial expansion to the expression

$$(\operatorname{Im} \Sigma)^{2k} = (2i)^{-2k} (\Sigma - \bar{\Sigma})^{2k},$$

we reduce the integral to a sum of integrals of the form

$$\int_T^{2T} \Sigma^{2k-v} \bar{\Sigma}^v dt \quad \text{for } v = 0, 1, \dots, 2k.$$

By (3.8), this integral will have no main term if $v \neq k$. Hence, by (3.7)

* The idea of this lemma is due to Selberg.

