THE DISTRIBUTION OF THE VALUES OF THE RIEMANN ZETA-FUNCTION

Kai-man Tsang

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Abstract

The Distribution of the Values of the Riemann Zeta-Function

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This thesis consists of two chapters devoted to the study of the distribution of the values of $\zeta(s)$, the Riemann's zeta-function.

Chapter one deals with the estimation of integrals of the form $\int_T^{2T} F\{\log\zeta(\sigma+it)\}dt$, where $T \to \infty$, $\sigma \in [\frac{1}{2},1]$ and $F(z)$ belongs to a certain class of functions. In particular, we derive asymptotic formulae for the integrals: $\int_T^{2T} \operatorname{sgn}\{S(t) - \alpha\}dt$, $\int_T^{2T} |\log|\zeta(\sigma+it)||dt$ and $\int_T^{2T} \operatorname{sgn}\{S(t)\}\operatorname{sgn}\{S(t+h)\}dt$, where α, h are positive and $S(t) := \pi^{-1}\operatorname{Im}\log\zeta(\frac{1}{2}+it)$. These results have interesting consequences. For example, we proved that : for $t \in [T,2T]$, $(\pi\log\log t)^{-\frac{1}{2}}\log\zeta(\frac{1}{2}+it)$ has $(0,1)$ Gaussian distribution in the complex plane. We obtained upper and lower estimates for the number of sign changes of $S(t)$ in the interval $[T,2T]$. We also deduced some results about the distribution of the zeros of $\zeta(s) - a$, where a is a fixed non-zero complex number.

Chapter two is a study of the extreme values of $\log\zeta(s)$ inside the critical strip. By refining a method of Selberg, we were able to improve some Ω-results about $\log\zeta(s)$. For instance, we proved that $S(t) = \Omega_{\pm}\{(\log t/\log\log t)^{1/3}\}$, $S_1(t) = \Omega_{-}\{(\log t)^{1/3}(\log\log t)^{-4/3}\}$ and $S_1(t) = \Omega_{+}\{(\log t)^{\frac{1}{2}}(\log\log t)^{-9/4}\}$. The last one is particularly interesting because it has come very close to what we can obtained under the Riemann Hypothesis.

We also extend our study to the functions $S(t+h) - S(t)$ and $S_1(t+h) - S_1(t)$. We proved that $\sup_{t\in[T,2T]} \pm\{S(t+h) - S(t)\} \geq c(h\log T)^{1/3}$ and $\sup_{t\in[T,2T]} \pm\{S_1(t+h) - S_1(t)\} \geq ch(\log T/\log\log T)^{1/3}$, for any fixed h in the interval $[(\log T)^{-1}, (\log\log T)^{-1}]$.

Finally, we studied the extreme values of the function $S(\sigma,t)$ (being defined as $\pi^{-1}\mathrm{Im}\log\zeta(\sigma+it)$) and obtained some Ω-theorems for $S(\sigma,t)$ when $\sigma - \frac{1}{2}$ is a positive decreasing function of t.

NOTATION

Throughout this thesis, we use j, m, n, k (with or without subscripts) to denote positive (or non-negative) integers and use p, q in the same way to denote primes. As usual, **R** is the set of all real numbers and **C** is the set of all complex numbers. For any $z \in$ **C**, **Re**z, **Im**z, $\bar{z}$ and $|z|$ are respectively the real part, imaginary part, conjugate and absolute value of z. Furthermore, [x] denotes the integral part of x.

The letters c and c_1, c_2,... etc. will be used exclusively to denote unspecified (usually absolute) positive constants which need not be the same at each occurrence.

The symbols $\ll$, $\gg$, o, O, Ω, Ω_+, Ω_- and $\Omega_\pm$ will have the usual meaning. For example, we write $f(x) = O(g(x))$ to mean that $|f(x)| \le cg(x)$ for all values of x under consideration. We also write $f(x) \sim g(x)$ when $\lim \{f(x)/g(x)\} = 1$ as x tends to some limit (usually positive infinity).

For a function f(z), we use $\hat{f}$, f' to denote its Fourier transform and derivative respectively.

As usual, $s = \sigma+it$ denotes a complex variable and $\rho = \beta+i\gamma$ denotes the generic non-trivial zero of $\zeta(s)$ (in §8, $\rho = \beta+i\gamma$ denotes a-value of $\zeta(s)$). For $\sigma \in [0,1]$, $N(\sigma,t)$ is the number of zeros in the rectangle:

$\sigma < \beta < 1$, $0 < \gamma \le t$, and $N(t) = N(0,t)$. Riemann Hypothesis is abbreviated to **RH**. Usually, T is a large positive number and we abbreviate $\underbrace{\log\ldots\log T}_{n\text{-fold}}$ to $\pounds_n$ for n = 2, 3 and 4.

We shall use freely the following well known functions:

(i) $\Gamma(z)$, the Gamma function.

(ii) $n!$, the factorial of n.

(iii) $\binom{z}{n}$, the binomial coefficient, defined as $z(z-1)\ldots(z-n+1)/n!$.

(iv) $J_0(z)$, the Bessel function of order zero.

(v) $\mathrm{sgn}(x)$, the signum function.

0. Some Basic Estimates

We collect here some basic estimates which we shall use from time to time without explicit reference.

(1) If $z = x + iy$ and $|\arg z| \leq \pi - \delta$, δ fixed, then

$$\log\Gamma(z) = (z - \tfrac{1}{2})\log z - z + \tfrac{1}{2}\log 2\pi + O(|z|^{-1}) \quad \text{as} \quad |z| \to \infty.$$

For any $x \geq 1$, there exist c_1, c_2 such that

$$(c_1 x)^x \leq \Gamma(x) \leq (c_2 x)^x .$$

(2) If $\pi(x)$ is the number of primes not exceeding x, then

$$\pi(x) = \mathrm{li}(x) + O\{x \exp(-c\sqrt{\log x})\},$$

where

$$\mathrm{li}(x) := \int_2^x \frac{du}{\log u} = x/\log x + O(x/\log^2 x). \tag{0.1}$$

(3) For any large y,

$$\sum_{p\le y} p^{-1} = \log\log y + c + O(1/\log y). \tag{0.2}$$

$$\sum_{p\le y} p^{-\lambda} = \begin{cases} -\log(\lambda-1) + O(1) & \text{for } (1-\lambda)\log y < -1,\ \lambda < c, \quad (0.3) \\ \log\log y + O(1) & \text{for } |(\lambda-1)\log y| \le 1, \quad (0.4) \\ \log\log y + y^{1-\lambda}/\{(1-\lambda)\log y\} + O\{y^{1-\lambda}/\{(1-\lambda)\log y\}^2\} & \\ & \text{for } (1-\lambda)\log y > 1. \end{cases}$$

$$\sum_{p> y} p^{-\lambda} = \begin{cases} -\log\{(\lambda-1)\log y\} + O(1) & \text{for } 0 < (\lambda-1)\log y \le 1, \\ y^{1-\lambda}/\{(\lambda-1)\log y\} + O\{y^{1-\lambda}/\{(\lambda-1)\log y\}^2\} & \quad (0.5) \\ & \text{for } (\lambda-1)\log y > 1. \end{cases}$$

$$\sum_{p\le y} p^{-1}\log p = \log y + O(1). \tag{0.6}$$

$$\sum_{p\le y} p^{-1}\cos(h\log p) = \log\{\min(h^{-1},\log y)\} + O(1) \quad \text{for } 0 < h < c. \tag{0.7}$$

(1) and (2) are well known results. See p.73 and p.113 of [2]. The estimates in (3) can be proved by using (2) and Stieltjes integration.

CHAPTER ONE

The General Distribution of The Values of $\zeta(s)$

1. Introduction

Ever since Riemann introduced his zeta-function into the study of the distribution of primes, the study of the properties of $\zeta(s)$ has proved a fertile field for research by many mathematicians. As was already realised by Riemann, the key to the deeper investigation of the distribution of the primes lies in the study of $\zeta(s)$ in the so called critical strip: $0 < \mathrm{Re}\ s < 1$. In particular, the distribution of its complex zeros in this region has the most profound arithmetical significance. However, unlike the situation elsewhere, inside the critical strip, $\zeta(s)$ is neither a Dirichlet series nor an Euler product. This renders the study of $\zeta(s)$ a formidable task.

In the early forties, Professor Selberg delved very deeply into the theory of $\zeta(s)$ and made important contributions. He obtained remarkable improvements and refinements over some of the results obtained previously by mathematicians in the Hardy-Littlewood era. Moreover, he introduced new ideas and techniques which opened up broad prospects for further research in the field.

In the early fifties, he achieved another breakthrough in his work on $\zeta(s)$. Based on his previous ideas, he developed an efficient method to estimate some integrals involving $\log\zeta(s)$. The results thus obtained have very interesting applications.

The function $\log\zeta(s)$ is far more complicated than $\zeta(s)$ itself. It has a branch point at every zero of $\zeta(s)$. The customary way to define $\log\zeta(s)$ is as follows. On the real axis it is real. For any $s = \sigma + it$ off the real axis, $\log\zeta(\sigma+it)$ is taken to be the value obtained by extending continuously its value at $s = 2$ along the polygonal line joining the points at 2, $2+it$ and $\sigma+it$. If, however, t is the ordinate of a zero of $\zeta(s)$, $\log\zeta(\sigma+it)$ is defined to be $\frac{1}{2}\lim_{\varepsilon\to 0^+}\{\log\zeta(\sigma+i(t+\varepsilon)) + \log\zeta(\sigma+i(t-\varepsilon))\}$.

$\log\zeta(s)$ carries more information than $\zeta(s)$ itself. For example, it was first proved by Von Mangoldt in 1905 that the function $N(t)$, the number of zeros $\rho = \beta + i\gamma$ lying inside the rectangle : $0 < \beta < 1$, $0 < \gamma \leq t$, is given by

$$N(t) = \vartheta(t) + S(t),^* \tag{1.1}$$

where

$$\vartheta(t) = (t/2\pi)\log(t/2\pi e) + 7/8 + O(t^{-1}) \tag{1.2}$$

and

$$S(t) = \pi^{-1}\operatorname{Im}\log\zeta(\tfrac{1}{2}+it) = O(\log t).$$

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* See [2, p.98 eqn.(1)].

The term $O(t^{-1})$ in (1.1) is differentiable. From this we see that, $S(t)$ determines the distribution of the zeros over intervals of bounded length. Moreover, the maximum order of $S(t)$ tells how well $N(t)$ is approximated by $\vartheta(t)$. We shall come to this question in chapter two. In general, the behavior of $\log\zeta(s)$ inside the critical strip has significant consequences.

The goal of Selberg's method is to estimate integrals of the form

$$\int_T^{2T} F\{\log\zeta(\sigma+it)\}dt,$$

where $T \to \infty$, $\frac{1}{2} \le \sigma \le 1$ and $F(z)$ belongs to a general class of functions. Among the $F(z)$ that have interesting applications are : $\operatorname{sgn}(\mathrm{Re}z)$, $\operatorname{sgn}(\mathrm{Im}z)$, $|\mathrm{Re}z|$ and $|\mathrm{Im}z|$. For example, with the first two functions, the results obtained lead to the conclusion that $\log\zeta(\sigma+it)$ has Gaussian distribution in the complex plane when σ is close to $\frac{1}{2}$. With the third function, we obtained precise information about the distribution of a-values of $\zeta(s)$. We shall discuss these details in the later sections.

His method is elegant and innovative. Unfortunately, it has never been published. There were several occasions where others independently discovered special cases of the theory. During our regular discussion sessions, he indicated to me the basic ideas of his method and went through some of the more tricky arguments. My original thesis plan included the materials in chapter two and a study of the distribution of the square free numbers. I hope the inclusion of this chapter can help to make this beautiful theory available to other people in this field.

The details of the proofs presented here are my own and may not be the same as Selberg's original version. However, except for a slight sharpening in Theorem 7.1, the results proved here are essentially those obtained by Selberg.

2. The Ideas of The Method

In this section, we shall give a brief description of the key ideas in estimating integrals of the form

$$\int_{T}^{T+H} F\{\log\zeta(\sigma+it)\}\, dt,$$

where $T \to \infty$, $T^{\eta} < H \leq T$, $\frac{1}{2} < \eta$ and $\sigma \in [\frac{1}{2},1]$. Among the $F(z)$ that have interesting consequences are: $\mathrm{sgn}(\mathbf{Re}z)$, $\mathrm{sgn}(\mathbf{Im}z)$, $|\mathbf{Re}z|$ and $|\mathbf{Im}z|$.

The key ideas are easily conveyed by a specific case. For the sake of concreteness and simplicity, we consider the integral

$$\int_{T}^{T+H} \mathrm{sgn}\{\mathbf{Im}\log\zeta(\sigma+it)\}\, dt. \tag{2.1}$$

First of all (see Lemma 4.1),

$$\mathrm{sgn}(u) = F_{\Omega}(u) + O\{\sin^2(\pi\Omega u)/(\pi\Omega u)^2\} \quad \text{for } u \in \mathbf{R}, \tag{2.2}$$

where $\Omega > 0$,

$$F_\Omega(u) = \int_0^\Omega G(v/\Omega)\ \sin(2\pi uv)\ \frac{dv}{v},$$

and $G(v)$ is a nice function satisfying $0 \le G(v) \le 1$ for $|v| \le 1$.

This function $F_\Omega(u)$ has two nice features.

(1)

$$F_\Omega(u) = P_N(u) + O\{(2\pi\Omega u)^{2N}/(2N)!\} \quad \text{for} \quad u \in \mathbf{R},$$

where $P_N(u)$ is a polynomial of degree at most $2N-1$. This follows from the expansion

$$\sin x = \sum_{0 \le n < N} (-1)^n x^{2n+1}/(2n+1)! \; + \; O\{x^{2N}/(2N)!\} \quad \text{for} \quad x \in \mathbf{R}. \qquad (2.3)$$

The key point here is that the error term has a very small coefficient. Consequently, when u is in a moderate range and Ω is of lower order than N, the error term is negligible and then $F_\Omega(u)$ is essentially a polynomial.

(2)

$$F_\Omega(u) = \mathbf{Im} \int_0^\Omega G(v/\Omega) e^{2\pi iuv} \frac{dv}{v}. \qquad (2.4)$$

Roughly speaking, $F_\Omega(u)$ behaves like $e^{2\pi iuv}$ in the sense that when u is a sum, $F_\Omega(u)$ becomes a product. The significance of this will be evident in step 2 described below.

The parameter Ω here is usually large. It helps to make the O-term in (2.2) small for most values of u unless Ωu is small.