$$= H(J_1 + J_2 + J_3) + O\{H/\psi^2(T)\}, \tag{4.42}$$

where

$$J_j := \int_0^\Omega \int_0^\Omega G(v/\Omega)G(w/\Omega)\Phi_j(v,w)\,\frac{dv}{v}\,\frac{dw}{w} \quad \text{for } j = 1,\ 2,\ 3.$$

We shall now use the previous three lemmas to show that J_2 and J_3 constitute remainder terms of lower order.

By definition of $\Phi_2(v,w)$ and (4.38),

$$J_2 = \tfrac{1}{2}\int_0^\Omega \int_0^\Omega G(v/\Omega)G(w/\Omega)\{P(v,-w) - P(v,w)\}\{\sum_{k=2}^{\infty} \sum_{\substack{m+n=2k\\ m,n\geq 0}} u_{m,n}v^m(-w)^n\}\,\frac{dv}{v}\,\frac{dw}{w}$$

$$\ll \sum_{2\leq k\leq K} + \sum_{K<k}\{\sum_{\substack{m+n=2k\\ m,n\geq 0}} |u_{m,n}|\int_0^\Omega \int_0^\Omega v^{m-1}w^{n-1}\{P(v,-w) - P(v,w)\}dvdw\},$$

where

$$K := \psi^{12}(T) \sim (\log\log T)^{12}. \tag{4.43}$$

Therefore, by Lemma 4.7 and (4.39),

$$J_2 \ll \sum_{2\leq k\leq K} k\{c\psi(2k)/2k\}^k\{c2k/\eta(y)\}^k\{\eta(y)/\psi(y)\}^{\frac{1}{2}} + \sum_{K<k} k\{c\psi(2k)/2k\}^k\Omega^{2k}$$

$$\ll \{\eta(y)/\psi(y)\}^{\frac{1}{2}} \sum_{2\leq k\leq K}\{c\psi(2k)/\eta(y)\}^k + \sum_{K<k}\{c\psi(2k)\Omega^2/k\}^k.$$

In the first sum, by (4.36),

$$c\psi(2k)/\eta(y) \leq c\pounds_4/\pounds_3^a = o(1).$$

Therefore the first sum is

$$<< \{\eta(y)/\psi(y)\}^{\frac{1}{2}}/\eta^2(y) << (\log R)^{-3/2}(\pounds_2)^{-\frac{1}{2}}.$$

In the second sum, by (4.43),

$$\Omega^2\psi(2k)/k < \Omega^2/\sqrt{k} \le \Omega^2/\sqrt{K} = \Omega^2/\psi^6(T) = \{\psi(T)\}^{-2}.$$

Therefore the second sum is negligible. Thus, we have

$$J_2 << (\log R)^{-3/2}(\pounds_2)^{-\frac{1}{2}}. \tag{4.44}$$

As regards J_3, we have

$$J_3 = \tfrac{1}{2}\int_0^\Omega\int_0^\Omega G(v/\Omega)G(w/\Omega)P(v,w)\{\sum_{k=2}^\infty \sum_{\substack{m+n=2k\\ m,n\ge 0}} u_{m,n}v^m((-w)^n - w^n)\}\ \frac{dv}{v}\frac{dw}{w}$$

$$<< \sum_{k=2}^\infty \sum_{\substack{m+n=2k\\ m,n\ge 0\\ m,n=odd}} |u_{m,n}|\int_0^\Omega\int_0^\Omega P(v,w)v^{m-1}w^{n-1}dvdw.$$

Similar to the argument for J_2, we divide this sum into two parts and use (4.39) to bound the coefficients $u_{m,n}$. The integrals are simpler and are bounded by Lemma 4.8 in an obvious way. The result is

$$J_3 << \psi(T)^{-2}. \tag{4.45}$$

Finally, we come to the estimation of the main term J_1. For any $r > b > 0$ and any $\xi,\ \eta\ \varepsilon\ \mathbf{C}$, repeated integrations show that

$$\frac{1}{\pi\lambda}\int_{-\infty}^\infty\int_{-\infty}^\infty \exp\{-\lambda^{-2}r(x^2+u^2) + 2\lambda^{-2}bxu\}e^{2\pi i(x\xi+u\eta)}dxdu$$

$$= \exp\{-\pi^2 r(\xi^2+\eta^2) - 2\pi^2 b\xi\eta\}, \tag{4.46}$$

where $\lambda = (r^2 - b^2)^{\frac{1}{2}}$.

From Lemma 4.1, we have

$$F_\Omega(x)F_\Omega(u) = \int_0^\Omega\int_0^\Omega G(v/\Omega)G(w/\Omega)\sin(2\pi xv)\sin(2\pi uw)\frac{dv}{v}\frac{dw}{w}$$

$$= \tfrac{1}{2}\mathrm{Re}\int_0^\Omega\int_0^\Omega G(v/\Omega)G(w/\Omega)\{e^{2\pi i(xv-uw)} - e^{2\pi i(xv+uw)}\}\frac{dv}{v}\frac{dw}{w}.$$

Define temporarily

$$\lambda := \{\psi^2(y) - \tau^2(y)\}^{\frac{1}{2}}. \tag{4.47}$$

Multiply both sides of the above equation by

$$\frac{1}{\pi\lambda}\exp\{-\lambda^{-2}(x^2+u^2)\psi(y) + 2\lambda^{-2}xu\tau(y)\}$$

and then integrate with respect to x and u over $\mathbf{R}^2$, the left side becomes

$$\frac{1}{\pi\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F_\Omega(x)F_\Omega(u)\exp\{-\lambda^{-2}(x^2+u^2)\psi(y) + 2\lambda^{-2}xu\tau(y)\}dxdu.$$

By (4.46) and the definition of P(v,w), we see that the right side is

$$\tfrac{1}{2}\int_0^\Omega\int_0^\Omega G(v/\Omega)G(w/\Omega)\{P(v,-w) - P(v,w)\}\frac{dv}{v}\frac{dw}{w}$$

$$= \int_0^\Omega\int_0^\Omega G(v/\Omega)G(w/\Omega)\Phi_1(v,w)\frac{dv}{v}\frac{dw}{w} = J_1.$$

Thus, in view of (4.2),

$$J_1 = \frac{1}{\pi\lambda}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\text{sgn}(x)\text{sgn}(u)\exp\{-\lambda^{-2}(x^2+u^2)\psi(y) + 2\lambda^{-2}xu\tau(y)\}dxdu$$

$$+ O\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\pi\Omega x)^{-2}\sin^2(\pi\Omega x)\exp\{-\lambda^{-2}(x^2+u^2)\psi(y) + 2\lambda^{-2}xu\tau(y)\}dxdu\}$$

$$= \frac{2}{\pi\lambda}\int_0^{\infty}\int_0^{\infty}\{\exp\{-\lambda^{-2}(x^2+u^2)\psi(y) + 2\lambda^{-2}xu\tau(y)\}$$

$$- \exp\{-\lambda^{-2}(x^2+u^2)\psi(y) - 2\lambda^{-2}xu\tau(y)\}\}dxdu$$

$$+ O\{\int_0^{\infty}(\pi\Omega x)^{-2}\sin^2(\pi\Omega x)\exp\{-\lambda^{-2}x^2\psi(y)\}\int_0^{\infty}\exp\{-\lambda^{-2}u^2\psi(y) + 2\lambda^{-2}xu\tau(y)\}dudx\}$$

$$= \frac{2}{\pi\lambda}\int_0^{\frac{\pi}{2}}\int_0^{\infty}\{\exp\{-\lambda^{-2}r^2(\psi(y) - \tau(y)\sin2\phi)\}$$

$$- \exp\{-\lambda^{-2}r^2(\psi(y) + \tau(y)\sin2\phi)\}\}rdrd\phi$$

$$+ O\{(\lambda/\sqrt{\psi(y)})\int_0^{\infty}(\pi\Omega x)^{-2}\sin^2(\pi\Omega x)\exp\{-\lambda^{-2}x^2(\psi(y) - \tau^2(y)/\psi(y))\}dx\}$$

$$= \frac{\lambda}{\pi}\int_0^{\frac{\pi}{2}}\{(\psi(y) - \tau(y)\sin2\phi)^{-1} - (\psi(y) + \tau(y)\sin2\phi)^{-1}\}d\phi$$

$$+ O\{(\lambda/\sqrt{\psi(y)})\int_0^{\infty}(\pi\Omega x)^{-2}\sin^2(\pi\Omega x)\exp\{-x^2/\psi(y)\}dx\}$$

$$= \frac{2}{\pi}\arctan(\tau(y)/\lambda) + O\{(\lambda/\sqrt{\psi(y)})\int_0^{\infty}(\pi\Omega x)^{-2}\sin^2(\pi\Omega x)dx\}$$

$$= 1 - \frac{2\lambda}{\pi\tau(y)} + O\{\lambda^3/\tau^3(y)\} + O\{\lambda/(\Omega\sqrt{\psi(y)})\}.$$

From (4.47),

$$\lambda = \{2\psi(y)\eta(y)\}^{\frac{1}{2}}\{1 + O\{\eta(y)/\psi(y)\}\}$$

$$= \{2\psi(y)\eta(y)\}^{\frac{1}{2}} + O\{\{\eta^3(y)/\psi(y)\}^{\frac{1}{2}}\}.$$

Hence

$$J_1 = 1 - \frac{2}{\pi}\{2\eta(y)/\tau(y)\}^{\frac{1}{2}} + O\{\{\eta(y)/\psi(y)\}^{3/2}\}. \qquad (4.48)$$

Finally, collecting the estimates (4.48), (4.45) and (4.44), we deduce from (4.42) that

$$\int_T^{T+H} \operatorname{sgn}\{\operatorname{Im}\Sigma_y(t)\}\operatorname{sgn}\{\operatorname{Im}\Sigma_y(t+h)\}dt = H - \frac{2H}{\pi}\{2\eta(y)/\tau(y)\}^{\frac{1}{2}}$$

$$+ O\{H(\log R)^{-3/2}(\log\log T)^{-\frac{1}{2}}\}.$$

Thus we have proved

Lemma 4.10. *Let* $T \to \infty$, $T^{\frac{1}{2}} < H \leq T$, $N = [\psi^6(T)]$ *and* $y = H^{1/(4N)}$. *Suppose* $h = R/\log T$, $R = \exp\{(\log\log\log T)^a\}$ *and* $1 < a \leq 3$, *then*

$$\int_T^{T+H} \operatorname{sgn}\{\operatorname{Im}\Sigma_y(t)\}\operatorname{sgn}\{\operatorname{Im}\Sigma_y(t+h)\}dt = H - \frac{2H}{\pi}(2\log R\ /\log\log T)^{\frac{1}{2}}$$

$$+ O\{H(\log\log\log T)(\log R\ \log\log T)^{-\frac{1}{2}}\}.$$

5. The Approximation of $\log\zeta(s)$

We mentioned in §2 that $\Sigma_y(\sigma,t)$ provides a good approximation to $\log\zeta(\sigma+it)$ for $\sigma \le 1$. In this section, we shall prove this in a more precise form. Recall from §2 ((2.5)) the definition

$$r_y(\sigma,t) = \log\zeta(\sigma+it) - \Sigma_y(\sigma,t) \quad \text{for} \quad y \ge 3, \quad \sigma \in [\tfrac{1}{2},1].$$

Define similarly,

$$r_y^*(\sigma,t) := \log\zeta(\sigma+it) - \Sigma_y^*(\sigma,t) \quad \text{for } y \ge 3, \quad \sigma \in [\tfrac{1}{2},1].$$

Our main result is

THEOREM 5.1. *Let* $T \to \infty$, $T^\eta < H \le T$, $\eta > \frac{1}{2}$ *and* $\sigma \in [\frac{1}{2},1]$. *If* k *is a positive integer,* $\varepsilon < (\eta-\frac{1}{2})/48$ *is a fixed small positive number and* $x = T^{\varepsilon/k}$, *then*

$$\int_T^{T+H} \{\operatorname{Im} r_x(\sigma,t)\}^{2k} dt = O\{H(ck)^{2k}\} \tag{5.1}$$

and

$$\int_T^{T+H} \{\mathrm{Re}\, r_x(\sigma,t)\}^{2k} dt = O\{H(ck)^{4k}\}. \tag{5.2}$$

Corollary. *If* $3 \le y \le x$, *we have*

$$\int_T^{T+H} \{\mathrm{Im}\, r_y(\sigma,t)\}^{2k} dt = O\{Hc^k\{k^{2k} + k^k(\sum_{y<p\le x} p^{-2\sigma})^k\}\} \tag{5.3}$$

and

$$\int_T^{T+H} \{\mathrm{Re}\, r_y(\sigma,t)\}^{2k} dt = O\{Hc^k\{k^{4k} + k^k(\sum_{y<p\le x} p^{-2\sigma})^k\}\}.$$

Furthermore, we have

$$\int_T^{T+H} \{\mathrm{Im}\, r_y^*(\sigma,t)\}^{2k} dt = O\{Hc^k\{k^{2k} + k^k(\sum_{y<p\le x} p^{-2\sigma})^k\}\} \tag{5.4}$$

and

$$\int_T^{T+H} \{\mathrm{Re}\, r_y^*(\sigma,t)\}^{2k} dt = O\{Hc^k\{k^{4k} + k^k(\sum_{y<p\le x} p^{-2\sigma})^k\}\}. \tag{5.5}$$

The corollary is an easy consequence of the theorem. For example,

$$\int_T^{T+H} \{\mathrm{Im}\, r_y(\sigma,t)\}^{2k} dt \le 2^{2k}\{\int_T^{T+H} \{\mathrm{Im}\, r_x(\sigma,t)\}^{2k} dt + \int_T^{T+H} |\sum_{y<p\le x} p^{-\sigma-it}|^{2k} dt\}.$$

The last integral, according to Lemma 3.3 (especially (3.10) and (3.7)), is

$$\le (H + x^k)(ck)^k(\sum_{y<p\le x} p^{-2\sigma})^k << H(ck)^k(\sum_{y<p\le x} p^{-2\sigma})^k.$$

Hence (5.3) follows immediately from (5.1).

Remark 1. Actually, the proof of the theorem shows that we can replace the O-term in (5.1) and (5.2) by $O\{H(ck)^{2k}(H/\sqrt{T})^{\frac{1}{16}(\frac{1}{2}-\sigma)}\}$ and $O\{H(ck)^{4k}(H/\sqrt{T})^{\frac{1}{32}(\frac{1}{2}-\sigma)}\}$ respectively. We can take advantage of these more precise estimates when $\sigma - \frac{1}{2}$ is not too small.

Remark 2. The case $\sigma = \frac{1}{2}$ in (5.1) was first proved by Selberg in [14] without giving the explicit factor in k. Recently, Ghosh [4] obtained $O\{H(ck)^{4k}\}$. The slight improvement we obtained here comes from an extra saving in our Lemma 5.2 which is corresponding to his Lemma 2. This improves the final results in §§7 & 8 by a power of logloglogT.

Theorem 5.1 is a rather straight forward generalization of the special case proved by Selberg. (5.1) and (5.2) are proved by parallel arguments. At the final step (see Lemma 5.5), we have to estimate a certain sum involving the zeros of $\zeta(s)$. The estimation of its real part is more involved and we cannot do as good as for its imaginary part. This accounts for the worse result in (5.2). Indeed, we observe that

$$\mathbf{Re}\, r_x(\sigma,t) = \log|\zeta(\sigma+it)| - \mathbf{Re}\,\Sigma_x(\sigma,t)$$

tends to $-\infty$ as $\sigma + it$ comes close to a zero of $\zeta(s)$.

In addition to the assumptions in the theorem, we borrow the following notation from Selberg's paper [14].

For any $t \in [T,T+H]$, define

$$\sigma_{x,t} := \tfrac{1}{2} + 2\max\{\beta-\tfrac{1}{2}, 2/\log x\}, \tag{5.6}$$

where the maximum is taken over all zeros $\rho = \beta+i\gamma$ satisfying

$$|t-\gamma| \le x^{3(\beta-\frac{1}{2})}/\log x, \quad \beta \ge \tfrac{1}{2}.$$

We first prove some lemmas.

Lemma 5.1. *Let* $3 \le X \le (H/\sqrt{T})^{\frac{1}{4}}$. *For any non-negative real number* ν, *we have*

$$\sum_{\substack{\beta>\sigma \\ T<\gamma\le T+H}} (\beta-\sigma)^{\nu} X^{(\beta-\sigma)} = O\{H(H/\sqrt{T})^{\frac{1}{2}(\frac{1}{2}-\sigma)}(c\nu)^{\nu}(\log T)^{1-\nu}\}.$$

Proof. The essential ingredient of the proof is already contained in Lemma 12 of [14]. Let

$$\delta_{\nu} := \begin{cases} 1 & \text{if } \nu = 0, \\ 0 & \text{if } \nu > 0. \end{cases}$$

It is easily seen that

$$\sum_{\substack{\beta>\sigma \\ T<\gamma\le T+H}} (\beta-\sigma)^{\nu} X^{(\beta-\sigma)} = \sum_{\substack{\beta>\sigma \\ T<\gamma\le T+H}} \{\int_0^{\beta-\sigma} d(u^{\nu}X^{u}) + \delta_{\nu}\}$$

$$= \int_0^{1-\sigma} \sum_{\substack{\beta>\sigma+u \\ T<\gamma\le T+H}} (\nu u^{\nu-1}X^{u} + u^{\nu}X^{u}\log X)du + \delta_{\nu}\{N(\sigma,T+H) - N(\sigma,T)\}$$

$$= \int_0^{1-\sigma} \{N(\sigma+u,T+H) - N(\sigma+u,T)\}(\nu u^{\nu-1}X^{u} + u^{\nu}X^{u}\log X)du \quad +$$

$$+\ \delta_\nu\{N(\sigma,T+H)\ -\ N(\sigma,T)\}.$$

Selberg has proved that [14, Theorem 1]

$$N(\sigma,T+H)\ -\ N(\sigma,T)\ =\ O\{H(H/\sqrt{T})^{\frac{1}{2}(\frac{1}{2}-\sigma)}\log T\}\quad\text{for}\quad \sigma\ \varepsilon\ [\tfrac{1}{2},1]. \tag{5.7}$$

Hence, the above integral is

$$\ll\ H(H/\sqrt{T})^{\frac{1}{2}(\frac{1}{2}-\sigma)}(\log T)\{\nu\int_0^{1-\sigma}u^{\nu-1}(X^{-2}H/\sqrt{T})^{-\frac{1}{2}u}du\ +\ \log X\int_0^{1-\sigma}u^{\nu}(X^{-2}H/\sqrt{T})^{-\frac{1}{2}u}du\}$$

$$\ll\ H(H/\sqrt{T})^{\frac{1}{2}(\frac{1}{2}-\sigma)}(\log T)\{\{\tfrac{1}{2}\log(X^{-2}H/\sqrt{T})\}^{-\nu}\Gamma(\nu+1)$$

$$+\ \{\tfrac{1}{2}\log(X^{-2}H/\sqrt{T})\}^{-\nu-1}\Gamma(\nu+1)\log X\}$$

$$\ll\ H(H/\sqrt{T})^{\frac{1}{2}(\frac{1}{2}-\sigma)}(c\nu)^{\nu}(\log T)^{1-\nu}.$$

Evidently, the term $\delta_\nu\{N(\sigma,T+H)\ -\ N(\sigma,T)\}$ has the same upper bound. ∎

This lemma will also be used in chapter two. An easy consequence of this is

Lemma 5.2. *Suppose* $3\ \le\ x^3X^2\ \le\ (H/\sqrt{T})^{\frac{1}{4}}$. *For any non-negative real number* ν, *we have*

$$\int_{\substack{\sigma_{x,t}>\sigma\\ T<t\le T+H}}(\sigma_{x,t}-\ \sigma)^{\nu}X^{(\sigma_{x,t}-\sigma)}dt\ =\ O\{H(H/\sqrt{T})^{\frac{1}{2}(\frac{1}{2}-\sigma)}c^{\nu+k}(\nu+k)^{\nu}(\log T)^{-\nu}\}.$$

Proof. By definition of $\sigma_{x,t}$, a crude estimate shows that the integral is

$$\leq \frac{2x^{3(\beta-\frac{1}{2})}}{\log x} \sum_{\substack{\beta > \frac{1}{2}(\sigma+\frac{1}{2}) \\ T-H<\gamma\leq T+2H}} \{\tfrac{1}{2}+2(\beta-\tfrac{1}{2})-\sigma\}^{\nu} X^{(\frac{1}{2}+2(\beta-\frac{1}{2})-\sigma)} + H(4/\log x)^{\nu} X^{4/\log x}.$$

The first term can be written as

$$\frac{2^{\nu} x^{\frac{3}{2}(\sigma-\frac{1}{2})}}{\log x} \sum_{\substack{\beta > \frac{1}{2}(\sigma+\frac{1}{2}) \\ T-H<\gamma\leq T+2H}} \{\beta-\tfrac{1}{2}(\sigma+\tfrac{1}{2})\}^{\nu} (x^3 X^2)^{(\beta-\frac{1}{2}(\sigma+\frac{1}{2}))}.$$

By Lemma 5.1 and the fact that $\log x = \frac{\varepsilon}{k}\log T$, this is

$$\ll Hk(H/\sqrt{T})^{\varepsilon(\frac{1}{2}-\sigma)}(c\nu)^{\nu}(\log T)^{-\nu}.$$

Clearly, the other term is $\ll Hc^{\nu+k}k^{\nu}(\log T)^{-\nu}$ if $\sigma \leq \frac{1}{2} + (4/\log x)$, and is zero otherwise. This completes the proof of the lemma. ∎

For $\sigma \in [\frac{1}{2},1]$, define

$$\lambda_t := \lambda(\sigma,x,t) = \max(\sigma_{x,t}, \sigma). \tag{5.8}$$

Lemma 5.3. *Let* $\{a_n\}$ *be a sequence of complex numbers such that*

(i) $a_n = 0$ *if* $\Lambda(n) = 0$,

(ii) *there is a positive constant* c *such that* $|a_{p^r}| \leq cr|a_p|$ *for all* p *and* $r \geq 2$.

Then, for any integer m, $k \leq m \leq 16k$, *we have*

$$\int_T^{T+H} \Big|\sum_{n<x^3} a_n n^{-\lambda\ -it}\Big|^{2m} dt = O\{H(cm)^m (\sum_{p\le x^3} |a_p|^2 p^{-2\sigma})^m\}.$$

Proof. First of all,

$$\sum_{n\le x^3} a_n n^{-\lambda_t -it} = \sum_{n\le x^3} a_n n^{-\sigma -it} + \sum_{n\le x^3} a_n (n^{-\lambda_t} - n^{-\sigma}) n^{-it}. \tag{5.9}$$

The first sum on the right decomposes into

$$\sum_{p\le x^3} a_p p^{-\sigma-it} + \sum_{p\le x^{3/2}} a_{p^2} p^{-2\sigma-2it} + \sum_{p\le x^3} \sum_{\substack{p^r\le x^3 \\ r\ge 3}} a_{p^r} p^{-r\sigma-rit}.$$

By assumption (ii), the double sum is

$$\ll \sum_{p\le x^3} |a_p| p^{-3\sigma} \ll (\sum_{p\le x^3} |a_p|^2 p^{-2\sigma})^{\frac{1}{2}},$$

by Cauchy's inequality and (0.3).

Hence

$$\int_T^{T+H} \Big|\sum_{n\le x^3} a_n n^{-\sigma-it}\Big|^{2m} dt \ll 3^{2m}\{\int_T^{T+H} \Big|\sum_{p\le x^3} a_p p^{-\sigma-it}\Big|^{2m} dt$$

$$+ \int_T^{T+H} \Big|\sum_{p\le x^{3/2}} a_{p^2} p^{-2\sigma-2it}\Big|^{2m} dt + H(\sum_{p\le x^3} |a_p|^2 p^{-2\sigma})^m\}.$$

By Lemma 3.3, the first integral on the right side is

$$\ll (H + x^{3m}) m! (\sum_{p\le x^3} |a_p|^2 p^{-2\sigma})^m \ll H(cm)^m (\sum_{p\le x^3} |a_p|^2 p^{-2\sigma})^m.$$

Similarly, the second integral has the same upper bound. Consequently, we have

$$\int_T^{T+H} \left|\sum_{n\le x^3} a_n n^{-\sigma-it}\right|^{2m} dt \ll H(cm)^m \Big(\sum_{p\le x^3} |a_p|^2 p^{-2\sigma}\Big)^m. \tag{5.10}$$

In view of (5.9), it remains to show that

$$\int_T^{T+H} \left|\sum_{n\le x^3} a_n (n^{-\lambda_t} - n^{-\sigma}) n^{-it}\right|^{2m} dt \ll H(cm)^m \Big(\sum_{p\le x^3} |a_p|^2 p^{-2\sigma}\Big)^m.$$

In fact, we shall prove the stronger estimate

$$\int_T^{T+H} \left|\sum_{n\le x^3} a_n (n^{-\lambda_t} - n^{-\sigma}) n^{-it}\right|^{2m} dt \ll H(cm/\log x)^m \Big(\sum_{p\le x^3} |a_p|^2 p^{-2\sigma}\log p\Big)^m. \tag{5.11}$$

Let $\xi = (H/\sqrt{T})^{1/(64m)}$ so that

$$\log\xi \asymp \frac{c}{k}\log T \quad \text{for some } c > 0. \tag{5.12}$$

Clearly,

$$\Sigma a_n (n^{-\lambda_t} - n^{-\sigma}) n^{-it} = -\int_\sigma^{\lambda_t} (\Sigma a_n \log n\, n^{-u-it}) du \ll \int_\sigma^{\lambda_t} |\Sigma a_n \log n\, n^{-u-it}| du.$$

For any $u \in [\sigma, \lambda_t]$,

$$\begin{aligned} |\Sigma a_n \log n\, n^{-u-it}| &= |\xi^u\, \Sigma a_n \log n\, (\xi n)^{-u} n^{-it}| \\ &= |\xi^u\, \Sigma a_n \log n\, n^{-it} \int_u^\infty (\xi n)^{-v} \log(\xi n)\, dv| \\ &\le \xi^{u-\sigma} \int_u^\infty \xi^{\sigma-v} |\Sigma a_n (\log n)(\log \xi n) n^{-v-it}| dv \\ &\le \xi^{\lambda_t - \sigma} \int_\sigma^\infty \xi^{\sigma-v} |\Sigma a_n (\log n)(\log \xi n) n^{-v-it}| dv. \end{aligned}$$

Hence

$$\Sigma a_n(n^{-\lambda_t} - n^{-\sigma})n^{-it} \ll (\lambda_t-\sigma)\xi^{\lambda_t-\sigma}\int_\sigma^\infty \xi^{\sigma-v}|\Sigma a_n(\log n)(\log\xi n)n^{-v-it}|dv.$$

Consequently,

$$\int_T^{T+H} |\sum_{n\le x^3} a_n(n^{-\lambda_t} - n^{-\sigma})n^{-it}|^{2m}dt$$

$$\ll \int_T^{T+H}(\lambda_t-\sigma)^{2m}\xi^{2m(\lambda_t-\sigma)}\{\int_\sigma^\infty \xi^{\sigma-v}|\sum_{n\le x^3} a_n(\log n)(\log\xi n)n^{-v-it}|dv\}^{2m}dt.$$

By Cauchy's inequality, this is

$$\ll \{\int_T^{T+H}(\lambda_t-\sigma)^{4m}\xi^{4m(\lambda_t-\sigma)}dt\}^{\frac{1}{2}}\times$$

$$\times\{\int_\sigma^\infty \xi^{\sigma-v}dv\}^{2m-\frac{1}{2}}\{\int_\sigma^\infty \xi^{\sigma-v}\int_T^{T+H} |\sum_{n\le x^3} a_n(\log n)(\log\xi n)n^{-v-it}|^{4m}dt\ dv\}^{\frac{1}{2}}.$$

By Lemma 5.2, the first integral is $\ll H(cm/\log T)^{4m}$, and by (5.10),

$$\int_T^{T+H}|\sum_{n\le x^3} a_n(\log n)(\log\xi n)n^{-v-it}|^{4m}dt$$

$$\ll H(cm)^{2m}\{\log x\ \log^2(\xi x)\}^{2m}(\sum_{p\le x^3}|a_p|^2p^{-2\sigma}\log p)^{2m}.$$

Collecting these estimates, we have

$$\int_T^{T+H}|\sum_{n\le x^3} a_n(n^{-\lambda_t} - n^{-\sigma})n^{-it}|^{2m}dt$$

$$\ll H(cm)^{3m}\{(\log\xi x)/\log\xi\}^{2m}(\log x\ /\log^2 T)^m(\sum_{p\le x^3}|a_p|^2p^{-2\sigma}\log p)^m.$$

In view of (5.12) and the fact that $\log x = \frac{\varepsilon}{k}\log T$, this last expression is

$$\ll H(cm/\log x)^m\Big(\sum_{p\le x^3}|a_p|^2p^{-2\sigma}\log p\Big)^m.$$

Our lemma is thus proved. ∎

We shall now follow the argument of Selberg in §4 of [14].

Define

$$\Lambda_x(n) := \begin{cases} \Lambda(n) & \text{for } 1 \le n \le x, \\ \Lambda(n)\times\frac{1}{2}\{\log^2(x^3/n) - 2\log^2(x^2/n)\}/\log^2 x & \text{for } x \le n \le x^2, \\ \Lambda(n)\times\frac{1}{2}\{\log^2(x^3/n)\}/\log^2 x & \text{for } x^2 \le n \le x^3. \end{cases} \tag{5.13}$$

Clearly,

$$|\Lambda_x(n)| \le \Lambda(n) \quad \text{for all} \quad n \ge 1.$$

Imitating Selberg's method, we can prove

Lemma 5.4. *For any* $t \in [T,T+H]$, $t \ne \gamma$ *and* $\sigma \in [\frac{1}{2},1]$, *we have*

$$\log\zeta(\sigma+it) = \begin{cases} \displaystyle\sum_{n\le x^3}\frac{\Lambda_x(n)}{\log n}n^{-\sigma-it} + O\{(\log x)^{-1}x^{\frac{1}{2}(\frac{1}{2}-\sigma)}\{|\sum_{n\le x^3}\Lambda_x(n)n^{-\sigma-it}| + \log T\}\} \\ \qquad\qquad \textit{for } \sigma \ge \sigma_{x,t}, \\ \displaystyle\sum_{n\le x^3}\frac{\Lambda_x(n)}{\log n}n^{-\sigma_{x,t}-it} + \end{cases}$$

$$+ O\{\{(\log x)^{-1}x^{\frac{1}{2}(\frac{1}{2}-\sigma_{x,t})} + (\sigma_{x,t}-\sigma)\}\{|\sum_{n\le x^3}\Lambda_x(n)n^{-\sigma_{x,t}-it}| + \log T\}\}$$

$$- \Sigma_\rho \int_\sigma^{\sigma_{x,t}} (\sigma_{x,t}-u)(u+it-\rho)^{-1}(\sigma_{x,t}+it-\rho)^{-1}du \quad \textit{for } \tfrac{1}{2} \le \sigma < \sigma_{x,t}. \quad (5.14)$$

Using the notation λ_t defined in (5.8), we can rewrite (5.14) as

$$\log\zeta(\sigma+it) - \sum_{p\le x} p^{-\sigma-it} = \sum_{p\le x} (p^{-\lambda_t} - p^{-\sigma})p^{-it}$$

$$+ \sum_{\substack{p^r\le x\\ r\ge 2}} r^{-1}p^{-r(\lambda_t+it)} + \sum_{x<n\le x^3} \frac{\Lambda_x(n)}{\log n} n^{-\lambda_t-it}$$

$$+ O\{\{(\log x)^{-1}x^{\frac{1}{4}-\frac{1}{2}\lambda_t} + (\lambda_t-\sigma)\}\{|\sum_{n\le x^3}\Lambda_x(n)n^{-\lambda_t-it}| + \log T\}\} - L(\sigma,t), \quad (5.15)$$

where

$$L(\sigma,t) = \Sigma_\rho \int_\sigma^{\lambda_t} (\lambda_t-u)(u+it-\rho)^{-1}(\lambda_t+it-\rho)^{-1}du.$$

Without much effort, we can deduce the following estimates from Lemmas 5.2 and 5.3.

(1) $\int_T^{T+H} |\sum_{p\le x} (p^{-\lambda_t} - p^{-\sigma})p^{-it}|^{2k}dt = O\{H(ck)^k\}.$

(2) $\int_T^{T+H} |\sum_{\substack{p^r\le x\\ r\ge 2}} r^{-1}p^{-r(\lambda_t+it)}|^{2k}dt = O\{H(ck)^k(\sum_{p\le\sqrt{x}} p^{-4\sigma})^k\}.$

(3) $\int_T^{T+H} |\sum_{x<n\le x^3} \frac{\Lambda_x(n)}{\log n} n^{-\lambda_t-it}|^{2k}dt = O\{H(ck)^k(\sum_{x<p\le x^3} p^{-2\sigma})^k\}.$

(4) $\int_T^{T+H} |\sum_{n\le x^3}\Lambda_x(n)n^{-\lambda_t-it}|^{4k}dt = O\{H(ck)^{2k}(\log x)^{4k}\}.$

(5) $\int_T^{T+H} x^{(\frac{1}{2}-\lambda_t)2k}dt = O\{Hx^{-2k(\sigma-\frac{1}{2})}\}.$