7. The Number of Sign Changes of $S(t)$

In 1935, Titchmarsh [15] proved that $S(t)$ * has an infinity of changes of sign. This was then improved by Selberg [14] in 1946. Among other things, he proved that $S(t)$ changes signs at least $T(\log T)^{1/3}\exp(-c\sqrt{\log\log T})$ times in the interval $(0,T)$. Later, he (unpublished) further improved this to $T(\log T)^{1-\varepsilon}$ for any $\varepsilon > 0$.

Let $\mathcal{T}(T,T+H)$ be the number of sign changes of $S(t)$ in the interval $[T,T+H]$. In this section, we shall make use of Theorems 6.1 and 6.2 to obtain upper and lower estimates for $\mathcal{T}(T,T+H)$. Our main result is

THEOREM 7.1. *Let* $T \to \infty$, $T^{\eta} < H \le T$ *and* $\eta > \frac{1}{2}$. *There exists* $c > 0$ *such that*

$$\mathcal{T}(T,T+H) \ge cH(\log T)\exp\{-c(\log\log\log T)^2\}, \; ** \qquad (7.1)$$

----

* $S(t) := \pi^{-1}\mathrm{Im}\log\zeta(\frac{1}{2}+it)$. See §1.

$$T(T,T+H) \le cH(\log T)(\log\log\log T)(\log\log T)^{-\frac{1}{2}}. \tag{7.2}$$

Proof. We first prove (7.1). Let

$$h = R/\log T \quad \text{and} \quad R = \exp(c_1 \mathcal{L}_3^2),$$

where c_1 is a large constant. With this h, Theorem 6.2 asserts that

$$\int_T^{T+H} \operatorname{sgn}\{S(t)\}\operatorname{sgn}\{S(t+h)\}dt = H - 2\pi^{-1}\sqrt{(2c_1)}H\mathcal{L}_3/\sqrt{\mathcal{L}_2} + O(H\mathcal{L}_3/\sqrt{\mathcal{L}_2})$$

$$\le H - (c_1)^{\frac{1}{4}}H\mathcal{L}_3/\sqrt{\mathcal{L}_2}, \tag{7.3}$$

when c_1 is sufficiently large.

Let $\nu = [H/h]$. Evidently,

$$\int_T^{T+h}\sum_{n=0}^{\nu-1} \operatorname{sgn}\{S(t+nh)\}\operatorname{sgn}\{S(t+(n+1)h)\}dt = \int_T^{T+H} \operatorname{sgn}\{S(t)\}\operatorname{sgn}\{S(t+h)\}dt + O(1).$$

Hence, by (7.3), there exists $t \in [T,T+h]$ such that

$$h\sum_{n=0}^{\nu-1} \operatorname{sgn}\{S(t+nh)\}\operatorname{sgn}\{S(t+(n+1)h)\} \le H - (c_1)^{\frac{1}{4}}H\mathcal{L}_3/\sqrt{\mathcal{L}_2} + O(1).$$

The sum on the left is equal to

** By a more refined argument, we can show that: *under the* **RH**, $T(T,T+H) \ge cH(\log T)\exp(-c\log\log\log T)$.

$$\nu - 2\times(\text{number of negative terms}) \geq \nu - 2T(T,T+H).$$

Thus

$$T(T,T+H) \geq \tfrac{1}{2}\nu - \frac{H}{2h}\{1 - (c_1)^{\frac{1}{4}}\pounds_3/\sqrt{\pounds_2}\} - O(h^{-1})$$

$$= \tfrac{1}{2}(c_1)^{\frac{1}{4}}H(\log T)\pounds_3\exp(-c_1\pounds_3^2)/\sqrt{\pounds_2} - O(h^{-1}) \geq cH(\log T)\exp(-c\pounds_3^2)$$

where $c = 2c_1$. This proves (7.1).

We now turn to the proof of (7.2). From the discussion in §1, we know that $S(t)$ decreases steadily when t is between two consecutive γ's and it jumps when t passes from γ^- to γ^+. In fact, if there are n zeros with the same γ (multiple zeros counted according to their multiplicities), then $S(\gamma^+) - S(\gamma^-) = n$. In case $n \geq 2$, we adopt the following convention. Let $\rho_j = \beta_j + i\gamma_j$, $j = 1, 2, \ldots, n$, be all the n zeros having the same ordinate γ. That is

$$\gamma_1 = \gamma_2 = \ldots = \gamma_n = \gamma.$$

We define

$$S(\gamma_j^+) = S(\gamma_{j+1}^-) \quad \text{for} \quad j = 1, 2, \ldots, n-1$$

so that under this convention,

$$S(\gamma_{j+1}) = S(\gamma_j) + 1 \quad \text{for} \quad j = 1, 2, \ldots, n-1.$$

(Note: $S(\gamma_j) = S(\gamma_j^+) - \frac{1}{2} = S(\gamma_j^-) + \frac{1}{2}$.)

It is easy to see that S(t) changes from negative to positive precisely at those jumps corresponding to $|S(\gamma)| \le \frac{1}{2}$. Of course, it changes from positive to negative at the same frequency. Therefore

$$T(T,T+H) = 2\times(\text{number of } \rho = \beta + i\gamma \text{ such that } |S(\gamma)| \le \tfrac{1}{2}) + O(1). \tag{7.4}$$

Let $\{\rho_j = \beta_j + i\gamma_j \mid j = 1, 2, \ldots, \nu\}$ be the sequence of zeros in the rectangle: $0 < \sigma < 1$, $T < t \le T+H$ such that

$$\gamma_1 \le \gamma_2 \le \ldots \le \gamma_\nu.$$

Since N(t), the zero counting function, is constant over the intervals $(\gamma_j^+, \gamma_{j+1}^-)$, we have, for any "reasonable" differentiable function f(x),

$$\int_{\gamma_j^+}^{\gamma_{j+1}^-} f'\{S(t)\}dN(t) = 0 \text{ for } j = 1, 2, \ldots, \nu-1. \tag{7.5}$$

Recall from §1 the formulas

$$N(t) = \vartheta(t) + S(t),$$

$$\vartheta(t) = (t/2\pi)\log(t/2\pi e) + 7/8 + O(t^{-1}).$$

We also know that

$$\vartheta'(t) = (1/2\pi)\log(t/2\pi) + O(t^{-2}).$$

Adding the equations in (7.5) for j = 1, 2,..., ν-1, we have

$$0 = \sum_{j=1}^{\nu-1} \int_{\gamma_j^+}^{\gamma_{j+1}^-} f'\{S(t)\}dN(t)$$

$$= \int_{\gamma_1^+}^{\gamma_\nu^-} f'\{S(t)\}d\vartheta(t) + \sum_{j=1}^{\nu-1} \int_{\gamma_j^+}^{\gamma_{j+1}^-} f'\{S(t)\}dS(t)$$

$$= (1/2\pi)\int_{\gamma_1^+}^{\gamma_\nu^-} \log(t/2\pi)\, f'\{S(t)\}dt + \sum_{j=1}^{\nu-1} \{f\{S(\gamma_{j+1}^-)\} - f\{S(\gamma_j^+)\}\}$$

$$+ O\{T^{-2}\int_{\gamma_1^+}^{\gamma_\nu^-} |f'\{S(t)\}|dt\}$$

$$= (1/2\pi)\log(T/2\pi)\int_{\gamma_1^+}^{\gamma_\nu^-} f'\{S(t)\}dt - \sum_{j=1}^{\nu} \{f\{S(\gamma_j^+)\} - f\{S(\gamma_j^-)\}\}$$

$$+ O\{\int_T^{T+H} |f'\{S(t)\}|dt + |f\{S(\gamma_\nu^+)\}| + |f\{S(\gamma_1^-)\}|\}.$$

Let

$$\Delta f(x) = f(x+\tfrac{1}{2}) - f(x-\tfrac{1}{2}) \text{ for } x \in \mathbf{R}.$$

The second sum on the right side above is

$$\sum_{j=1}^{\nu} \{f\{S(\gamma_j)+\tfrac{1}{2}\} - f\{S(\gamma_j)-\tfrac{1}{2}\}\} = \sum_{j=1}^{\nu} \Delta f\{S(\gamma_j)\}.$$

Hence

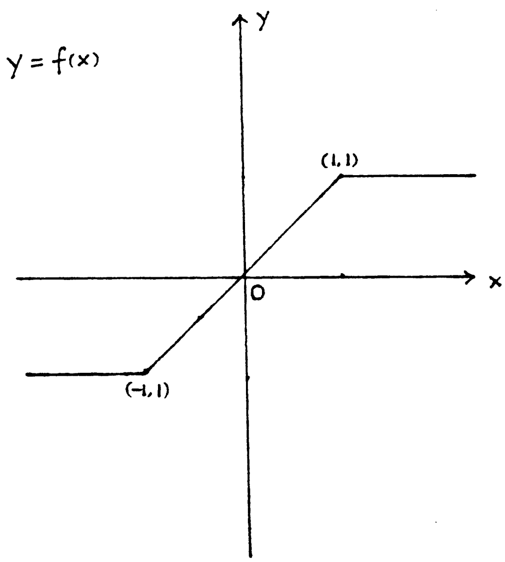
$$\sum_{j=1}^{\nu} \Delta f\{S(\gamma_j)\} = (1/2\pi)\log(T/2\pi)\int_{\gamma_1^+}^{\gamma_\nu^-} f'\{S(t)\}dt$$

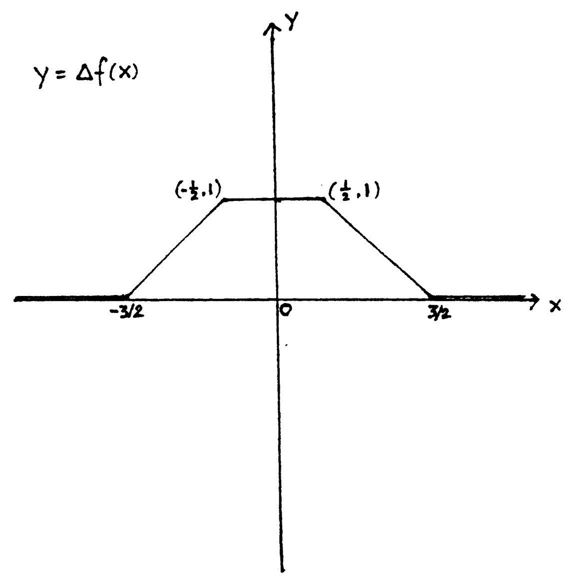
$$+ O\{\int_T^{T+H} |f'\{S(t)\}|dt + |f\{S(\gamma_\nu^+)\}| + |f\{S(\gamma_1^-)\}|\}. \quad (7.6)$$

We now take $f(x)$ to be the following function:

$$f(x) = \begin{cases} \operatorname{sgn}(x) & \text{if } |x| \geq 1, \\ x & \text{if } |x| < 1. \end{cases}$$

The graphs of $f(x)$ and $\Delta f(x)$ are:





For this particular $f(x)$, we see that $\Delta f(x)$ majorises $\chi_{-\frac{1}{2},\frac{1}{2}}(x)$, and $f'(x) = \chi_{-1,1}(x)$. * In view of (7.4) and (7.6), we have

$$T(T,T+H) \leq 2\sum_{j=1}^{\nu} \Delta f\{S(\gamma_j)\} = \frac{1}{\pi}\log(T/2\pi)\int_T^{T+H} \chi_{-1,1}\{S(t)\}dt + O(H).$$

Now we use Theorem 6.1 ((6.12)) to estimate this integral, the result is

----

* $f(x)$ is not differentiable at $x = \pm 1$. However, this does not cause any trouble because we can smooth the corners.

$$\int_T^{T+H} \chi_{-1,1}\{S(t)\}dt = H\int_{-(\pi/\mathcal{L}_2)^{\frac{1}{2}}}^{(\pi/\mathcal{L}_2)^{\frac{1}{2}}} e^{-\pi u^2}du + O(H\mathcal{L}_3/\sqrt{\mathcal{L}_2}) = O(H\mathcal{L}_3/\sqrt{\mathcal{L}_2}).$$

Hence

$$T(T,T+H) \le cH(\log T)\mathcal{L}_3/\sqrt{\mathcal{L}_2}$$

for a sufficiently large c. This proves (7.2). ∎

We notice that the upper estimate in (7.2) actually comes from the error term on the right of (6.12). If the asymptotic formula (6.12) were better, say

$$\int_T^{T+H} \chi_{\alpha,\alpha'}\{(\pi/\mathcal{L}_2)^{\frac{1}{2}}S(t)\}dt = H\int_{\alpha}^{\alpha'} e^{-\pi u^2}du + o(H\mathcal{L}_2^{-3/2}), \tag{7.7}$$

we can obtain an asymptotic estimate for $T(T,T+H)$. Indeed, by choosing an f so that Δf approximates $\chi_{-\frac{1}{2},\frac{1}{2}}$, we deduce from (7.6) and (7.7) that

$$\sum_{j=1}^{\nu} \chi_{-\frac{1}{2},\frac{1}{2}}\{S(\gamma_j)\} \sim \frac{1}{2\pi} H(\pi/\mathcal{L}_2)^{\frac{1}{2}}\log T.$$

In view of (7.4), we have the following

CONJECTURE.

$$T(T,T+H) \sim H(\pi\log\log T)^{-\frac{1}{2}}\log T.$$

§ 8. The Distribution of The a-values of $\zeta(s)$

Let a be a fixed non-zero complex number. We denote by $\rho = \beta + i\gamma$ * the generic zero of $\zeta(s) - a$, the so called a-values of $\zeta(s)$. Throughout this section, we assume

$$T \to \infty, \ T^{\eta} < H \le T \quad \text{and} \quad \eta > \tfrac{1}{2}.$$

For any $\sigma < \sigma'$, $M(\sigma,\sigma')$ denotes the number of ρ satisfying $\sigma < \beta \le \sigma'$ and $T < \gamma \le T + H$.

The function $M(\sigma,\sigma')$ has been studied extensively as a counterpart to the familiar function $N(\sigma,T)$. For example, it has been proved by Bohr and Jensen ** that

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* Except for a couple of occasions, $\rho = \beta + i\gamma$ is used in this section to denote a typical zero of $\zeta(s) - a$.

** See chapter 11 of [16] for more details.

$$M(\sigma,\sigma') \sim c(\sigma,\sigma')H$$

for $\frac{1}{2} < \sigma < \sigma' < 1$.

In this section, we shall make use of Theorems 6.1 and 6.3 to study $M(\sigma,\sigma')$ when σ is very close to $\frac{1}{2}$. We know that $\zeta(\sigma+it) \to 1$ uniformly in t as $\sigma \to \infty$. So, if $a \neq 1$, $M(\sigma,\infty)$ will be zero when σ is sufficiently large. On the other hand, the number of 1-values of $\zeta(s)$ in every rectangle: $\frac{1}{2} < \sigma < \beta < \sigma'$, $T < \gamma \le T + H$ is a multiple of H. So we consider two separate cases according to the distances between a and 0 or 1.

Our first theorem does not depend on the results in the previous sections.

THEOREM 8.1. *Let* $0 < \delta < 0.01$ *and* $\sigma \le 0$.

(1) *If* $\delta < |1-a| \le \delta^{-1}$, *then*

$$M(\sigma,\infty) = \frac{T+H}{2\pi}\log\frac{T+H}{2\pi} - \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{H}{2\pi} + O_\delta(\log T)$$

$$= \frac{H}{2\pi}\log\frac{T}{2\pi} + O_\delta(H^2T^{-1}). \tag{8.1}$$

(2) *If* $|1-a| \le \delta$, *then*

$$M(\sigma,4) = \frac{T+H}{2\pi}\log\frac{T+H}{2\pi} - \frac{T}{2\pi}\log\frac{T}{2\pi} - (1 + \log 2)\frac{H}{2\pi} + O_\delta(\log T)$$

$$= \frac{H}{2\pi}\log\frac{T}{2\pi} + O_\delta(H).$$

If RH were true, then both results remain valid for any fixed $\sigma < \frac{1}{2}$.

Proof. The proof makes use of the argument principle in a familiar way, so we only give a brief sketch.

Case (1) $\delta < |1-a| < \delta^{-1}$.

Let $\sigma_0 = 4 - 2\log\delta$. It is easy to check that in this case, $\zeta(s) - a$ has no zeros on the right of the line $\sigma = \sigma_0$. Let R be the rectangle with vertices at the points $\sigma + iT$, $\sigma_0 + iT$, $\sigma_0 + i(T+H)$ and $\sigma + i(T+H)$. Starting from $\sigma + iT$, we describe R counterclockwisely and denote its four sides by L_1, L_2, L_3 and L_4 respectively. We may assume that L_1 and L_3 do not contain any zero of $\zeta(s) - a$.

Apply argument principle to R, we have

$$2\pi M(\sigma,\infty) = 2\pi M(\sigma,\sigma_0) = \sum_{\nu=1}^{4} \Delta_{L_\nu} \arg\{\zeta(s) - a\}. \tag{8.2}$$

It is easy to see that

$$\Delta_{L_2} \arg\{\zeta(s) - a\} = O(1). \tag{8.3}$$

Apply Jensen's formula to circles centered at $\sigma_0 + iT$ and $\sigma_0 + i(T+H)$ respectively, we can show that *

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* Use the same type of argument which proves that $S(t) = O(\log t)$. See

$$\Delta_{L_1}\arg\{\zeta(s) - a\},\ \Delta_{L_3}\arg\{\zeta(s) - a\} = O_\delta(\log T). \tag{8.4}$$

It is well known that $\zeta(s)$ satisfies the following functional relation *:

$$\zeta(s) = \chi(s)\zeta(1-s) \quad \text{for all } s \in \mathbb{C},$$

where

$$\chi(s) = \pi^{s-\frac{1}{2}}\Gamma\{\tfrac{1}{2}(1-s)\}/\Gamma(\tfrac{1}{2}s).$$

When $|t|$ is large,

$$\chi(\sigma+it) = (2\pi/t)^{\sigma-\frac{1}{2}+it}e^{i(t+\frac{1}{4}\pi)}\{1 + O(|t|^{-1})\} \tag{8.5}$$

and

$$|\chi(\sigma+it)| \asymp (2\pi/t)^{\sigma-\frac{1}{2}}. \tag{8.6}$$

The major contribution to $M(\sigma,\infty)$ comes from $\Delta_{L_4}\arg\{\zeta(s) - a\}$. Indeed, due to the factor $\chi(s)$, $|\zeta(s)|$ is very large for $\sigma \leq 0$. Hence

$$\Delta_{L_4}\arg\{\zeta(s) - a\} = \Delta_{L_4}\arg\zeta(s) + O_\delta(1)$$

[6, p.69].

* [16, eqn.(2.1.9)].

$$= \Delta_{L_4} \arg\chi(s) + \Delta_{L_4} \arg\zeta(1-s) + O_\delta(1)$$

$$= \arg\chi(\sigma+iT) - \arg\chi(\sigma+i(T+H)) + O_\delta(\log T).$$

Combining this with (8.3), (8.4) and (8.2), we have

$$M(\sigma,\infty) = \frac{1}{2\pi}\{\arg\chi(\sigma+iT) - \arg\chi(\sigma+i(T+H))\} + O_\delta(\log T).$$

By (8.5), the first two terms on the right side give the main term in (8.1).

Case (2) $|1-a| \leq \delta$.

Take $\sigma_0 = 4$ and repeat the previous argument. The only difference here is that, on the line **Re** $s = 4$,

$$\zeta(s) - a = 1 - a + 2^{-4-it} + \sum_{n=3}^{\infty} n^{-s},$$

and 2^{-4-it} is the dominating term. Therefore

$$\Delta_{L_2} \arg\{\zeta(s) - a\} = \Delta_{L_2} \arg(2^{-4-it}) + O(1) = -H\log 2 + O(1).$$

The rest of the proof is the same as in case (1).

Finally, we notice that the condition $\sigma \leq 0$ is utilised to make sure that $|\zeta(\sigma+it)|$ is large for $t \in [T,T+H]$ when we estimate $\Delta_{L_4} \arg\{\zeta(s) - a\}$. Under the **RH**, this is still true for any fixed $\sigma < \frac{1}{2}$. This justifies the last statement of the theorem. ∎

Now we come to studying of the distribution of a-values in the vicinity of the critical line. We assume, for the rest of this section,

$$0 \le \sigma \le 1 \quad \text{unless specified otherwise.}$$

Similar to the situation in Theorem 8.1, we consider the two cases:

(1) $\delta < |1-a|$, $\delta < |a| \le \delta^{-1}$,

(2) $|1-a| \le \delta$.

Let

$$\sigma_0 = \begin{cases} \max(4 - 2\log\delta,\ 2\log T) & \text{for case (1),} \\ 4 & \text{for case (2).} \end{cases}$$

Denote by R the rectangle with vertices at the points $\sigma + iT$, $\sigma_0 + iT$, $\sigma_0 + i(T+H)$ and $\sigma + i(T+H)$. Starting from $\sigma + iT$, we describe R counterclockwisely and denote its four sides by L_1, L_2, L_3 and L_4 respectively. Since $\zeta(s) - a$ has no zeros on the line $\sigma = \sigma_0$, we can define in R the function $\log\{\zeta(s) - a\}$ in the same way as we define $\log\zeta(s)$. A familiar lemma of Littlewood* says that

$$M(v,\sigma_0)dv = (-1/2\pi i)\sum_{\nu=1}^{4}\int_{L_\nu}\log\{\zeta(s) - a\}\, ds. \tag{8.7}$$

----

* [16, p.187 eqn.(9.9.1)].

The integral on the left side is the sum

$$\sum_{\substack{\sigma<\beta<\sigma_0 \\ T<\gamma\le T+H}} (\beta-\sigma),$$

which is a real number. Therefore equation (8.7) is

$$\sum_{\substack{\sigma<\beta<\sigma_0 \\ T<\gamma\le T+H}} (\beta-\sigma) = \frac{1}{2\pi}\int_T^{T+H} \log|\zeta(\sigma+it) - a|dt - \frac{1}{2\pi}\int_T^{T+H} \log|\zeta(\sigma_0+it) - a|dt$$

$$+ \frac{1}{2\pi}\int_\sigma^{\sigma_0} \{\operatorname{Im}\log\{\zeta(\alpha+i(T+H)) - a\} - \operatorname{Im}\log\{\zeta(\alpha+iT) - a\}\}d\alpha.$$

The last integral is

$$-\Delta_{L_3}\arg\{\zeta(s) - a\} - \Delta_{L_1}\arg\{\zeta(s) - a\} = O_\delta(\log T),$$

by (8.4). So we have

$$\sum_{\substack{\sigma<\beta<\sigma_0 \\ T<\gamma\le T+H}} (\beta-\sigma) = \frac{1}{2\pi}(I_\sigma - I_{\sigma_0}) + O_\delta(\log T), \tag{8.8}$$

where

$$I_\sigma = \int_T^{T+H} \log|\zeta(\sigma+it) - a|dt \quad \text{and} \quad I_{\sigma_0} = \int_T^{T+H} \log|\zeta(\sigma_0+it) - a|dt.$$

The analysis of I_σ is rather elaborate. We shall do this in the next four lemmas.

Lemma 8.1. *For* $\frac{1}{2} \le \sigma \le 1$,

$$\int_T^{T+H} \log|\zeta(\sigma+it)|dt = \begin{cases} O\{H(H/\sqrt{T})^{\frac{1}{2}(\frac{1}{2}-\sigma)}\} & \textit{unconditionally}, \\ O(\log T) & \textit{under } \mathrm{RH}. \end{cases}$$

Proof. Equation (8.8) is valid for any complex number a. In particular, when a = 0, it becomes

$$\int_T^{T+H} \log|\zeta(\sigma+it)|dt = 2\pi \sum_{\substack{\sigma<\beta \\ T<\gamma\le T+H}} (\beta-\sigma) + O(\log T).$$

Here in this lemma, $\beta + i\gamma$ denotes ordinary zeros of $\zeta(s)$.

Under the RH, the sum is zero and the second assertion is evident. On the other hand, without any assumption, the proof that

$$\sum_{\substack{\sigma<\beta \\ T<\gamma\le T+H}} (\beta-\sigma) = O\{H(H/\sqrt{T})^{\frac{1}{2}(\frac{1}{2}-\sigma)}\}$$

for $\sigma \in [\frac{1}{2},1]$ is very difficult. It is equivalent to proving that

$$N(\sigma,T+H) - N(\sigma,T) = O\{H(H/\sqrt{T})^{\frac{1}{2}(\frac{1}{2}-\sigma)}\log T\}.$$

This celebrated theorem was first established by Selberg [14, Theorem 1] in 1946. ▮

Let

$$A = \min(1, \tfrac{1}{2}|a|) \quad \text{and} \quad B = \max(1, 2|a|),$$

so that $A \in [\frac{1}{2}\delta\ ,\ 1]$ and $B \in [1\ ,\ 2/\delta]$. Let*

$$P_1(\sigma) = \langle|\zeta(\sigma+it)|,\ 0,\ A\rangle,$$

$$P_2(\sigma) = \langle|\zeta(\sigma+it)|,\ B,\ \infty\rangle,$$

$$P_3(\sigma) = \langle|\zeta(\sigma+it)|,\ A,\ B\rangle.$$

Clearly, $P_1(\sigma) \cup P_2(\sigma) \cup P_3(\sigma) = [T,T+H]$.

For any $\sigma \in [\frac{1}{2},1]$ and any $\alpha' > \alpha$, it follows from (6.11) that

$$\|\langle|\zeta(\sigma+it)|,\ \alpha,\ \alpha'\rangle\| = H\int_{\eta}^{\eta'} e^{-\pi u^2}du + O\{H\{\log^2\psi(\sigma,T)\}\{\psi(\sigma,T)\}^{-\frac{1}{2}}\}, \qquad (8.9)$$

where $\eta = (\log\alpha)\{\pi\psi(\sigma,T)\}^{-\frac{1}{2}}$ and $\eta' = (\log\alpha')\{\pi\psi(\sigma,T)\}^{-\frac{1}{2}}$. In particular, if $\log\alpha$, $\log\alpha' \ll_\delta \log^2\psi(\sigma,T)$, then

$$\|\langle|\zeta(\sigma+it)|,\ \alpha,\ \alpha'\rangle\| = O_\delta\{H\{\log^2\psi(\sigma,T)\}\{\psi(\sigma,T)\}^{-\frac{1}{2}}\}. \qquad (8.10)$$

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* The general notation $\langle f(t),\alpha,\alpha'\rangle := \{t \in [T,T+H]\mid f(t) \in [\alpha,\alpha']\}$ has been defined in (6.1).

Lemma 8.2.

$$\int_{P_1(\sigma)} \log|\zeta(\sigma+it) - a|dt = \begin{cases} \frac{1}{2}H\log|a| + O_\delta\{H\{\log^2\psi(\sigma,T)\}\{\psi(\sigma,T)\}^{-\frac{1}{2}}\} & \\ & \textit{if} \quad \frac{1}{2} \le \sigma \le 1, \\ O_\delta(H) & \textit{if} \quad 0 \le \sigma < \frac{1}{2}. \end{cases} \tag{8.11}$$

Proof. (8.11) is evident because $\log|\zeta(\sigma+it) - a| = O_\delta(1)$ for $t \in P_1(\sigma)$.

Assume $\frac{1}{2} \le \sigma \le 1$. By (8.9) and (8.10),

$$\|P_1(\sigma)\| = \|<|\zeta(\sigma+it)|,\ 0,\ 1>\| - \|<|\zeta(\sigma+it)|,\ A,\ 1>\|$$

$$= \tfrac{1}{2}H - O_\delta\{H\{\log^2\psi(\sigma,T)\}\{\psi(\sigma,T)\}^{-\frac{1}{2}}\}. \tag{8.12}$$

For any $t \in P_1(\sigma)$,

$$\log|\zeta(\sigma+it) - a| = \log|a| + O_\delta(|\zeta(\sigma+it)|).$$

Hence, in view of (8.12),

$$\int_{P_1(\sigma)} \log|\zeta(\sigma+it) - a|dt = \tfrac{1}{2}H\log|a|$$

$$+ O_\delta\{H\{\log^2\psi(\sigma,T)\}\{\psi(\sigma,T)\}^{-\frac{1}{2}} + \int_{P_1(\sigma)} |\zeta(\sigma+it)|dt\ \}.$$

The integral in the remainder term can be broken up into

$$\int_{<0,\ \psi(\sigma,T)^{-1}>} + \int_{<\psi(\sigma,T)^{-1},\ A>} |\zeta(\sigma+it)|dt.$$