

A Note on the Differences Between Consecutive Primes.

Heath-Brown, D.R.; Goldston, D.A.

in: Mathematische Annalen | Mathematische Annalen | Periodical Issue | Article
317 - 320

Terms and Conditions

The Göttingen State and University Library provides access to digitized documents strictly for noncommercial educational, research and private purposes and makes no warranty with regard to their use for other purposes.

Some of our collections are protected by copyright. Publication and/or broadcast in any form (including electronic) requires prior written permission from the Goettingen State- and University Library.

Each copy of any part of this document must contain there Terms and Conditions. With the usage of the library's online system to access or download a digitized document you accept there Terms and Conditions.

Reproductions of material on the web site may not be made for or donated to other repositories, nor may be further reproduced without written permission from the Goettingen State- and University Library

For reproduction requests and permissions, please contact us. If citing materials, please give proper attribution of the source.

Contact:

Niedersächsische Staats- und Universitätsbibliothek

Digitalisierungszentrum

37070 Goettingen

Germany

Email: gdz@www.sub.uni-goettingen.de

Purchase a CD-ROM

The Goettingen State and University Library offers CD-ROMs containing whole volumes / monographs in PDF for Adobe Acrobat. The PDF-version contains the table of contents as bookmarks, which allows easy navigation in the document. For availability and pricing, please contact:

Niedersaechisische Staats- und Universitaetsbibliothek Goettingen - Digitalisierungszentrum

37070 Goettingen, Germany, Email: gdz@www.sub.uni-goettingen.de

A Note on the Differences Between Consecutive Primes

D. R. Heath-Brown¹ and D. A. Goldston²

¹ Magdalen College, Oxford OX1 4AU, England

² School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA

1. Introduction

The distribution of primes has recently been shown to be related to the pair correlation of the zeros of the Riemann Zeta-function $\zeta(s)$. The correlation is measured by the function

$$F(\alpha, T) = \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} \frac{4}{4 + (\gamma - \gamma')^2},$$

where α and $T \geq 2$ are real and γ, γ' run over the imaginary parts of the zeros of $\zeta(s)$. On the Riemann Hypothesis, Montgomery [4] proved that

$$F(\alpha, T) = (1 + o(1)) T^{-2\alpha} \log T + \alpha + o(1)$$

as $T \rightarrow \infty$, uniformly for $0 \leq \alpha \leq 1 - \varepsilon$, if $\varepsilon > 0$. Moreover he conjectured that $F(\alpha, T) \rightarrow 1$ as $T \rightarrow \infty$, uniformly for $1 \leq \alpha \leq M$, for any fixed $M > 1$. This would have important implications, both for the zeta-function and for the distribution of primes. In particular Heath-Brown [2], improving a result of Mueller [5], showed that Montgomery's conjecture, together with the Riemann Hypothesis, leads to

$$p_{n+1} - p_n = O(p_n^{1/2} (\log p_n)^{1/2}).$$

Here, only the fact that $F(\alpha, T) \ll 1$ was used. The above bound may be compared with the classical result of Cramér [1], that

$$p_{n+1} - p_n = O(p_n^{1/2} (\log p_n))$$

on the Riemann Hypothesis. The purpose of this note is to show how the full strength of Montgomery's conjecture can be applied to this problem.

Theorem. *Assume the Riemann Hypothesis. Suppose further that there exist a, b with $1 < a < 2 < b$ such that $F(\alpha, T) \rightarrow 1$ as $T \rightarrow \infty$, uniformly for $a \leq \alpha \leq b$. Then*

$$p_{n+1} - p_n = o(p_n^{1/2} (\log p_n)^{1/2}).$$

2. Proof of the Theorem

We assume the Riemann Hypothesis only for (1) below. It is not needed elsewhere. We first convert the problem into one depending on the distribution of the numbers γ . Write $p_{n+1} - p_n = \Delta$ and

$$S(T, x) = \left| \sum_{0 < \gamma \leq T} x^{i\gamma} \right|.$$

By the method of Mueller [5; Sect. 2] we find that there is a numerical constant C_0 such that if $\Delta \geq p_n^{1/2}$, then

$$S(T, x) \geq C_0 \Delta^2 T^2 p_n^{-3/2} \tag{1}$$

for some T, x satisfying $T \geq p_n/\Delta, p_n \leq x \leq p_{n+1}$. Since $S(T, x) \ll T \log T$ it is clear that (1) is false unless $T \ll p_n^{3/2} (\log p_n) \Delta^{-2}$, as we henceforth assume.

Our first lemma is a generalization of Heath-Brown [2; Lemma 4]. Let

$$\begin{aligned} G_\beta(x, T) &= \sum_{0 < \gamma, \gamma' \leq T} \frac{x^{i(\gamma - \gamma')}}{4\beta^2 + (\gamma - \gamma')^2} \\ &= \beta \int_{-\infty}^{\infty} e^{-2\beta|u|} \left| \sum_{0 < \gamma \leq T} x^{i\gamma} e^{i\gamma u} \right|^2 du. \end{aligned}$$

Lemma 1. *We have $S(T, x) \ll (T\beta^{-1} \max_{t \leq T} G_\beta(x, t))^{1/2}$ uniformly for $0 < \beta \leq T$ and $x \geq 1$.*

The proof is the same as in [2; Sect. 2] except that the extra parameter β is introduced by applying the Sobolev-Gallagher inequality [3; Lemma 1.1] in the form

$$|f(0)| \leq \frac{1}{2\delta} \int_{-\delta}^{\delta} |f(u)| du + \frac{1}{2} \int_{-\delta}^{\delta} |f'(u)| du,$$

with $\delta = 1/\beta$.

Lemma 2. *Write $G_1(x, t) = G(x, t)$. For $x > 0, \beta > 0, t \geq 2$, we have*

$$G_\beta(x, t) = \beta^2 G(x, t) + \beta(1 - \beta^2) \int_0^\infty G(u, t) \min \left\{ \left(\frac{u}{x} \right)^{2\beta}, \left(\frac{x}{u} \right)^{2\beta} \right\} \frac{du}{u}.$$

This follows immediately, on noting that

$$\int_0^\infty u^{iv} \min \left\{ \left(\frac{u}{x} \right)^{2\beta}, \left(\frac{x}{u} \right)^{2\beta} \right\} \frac{du}{u} = \frac{4\beta x^{iv}}{4\beta^2 + v^2}.$$

Lemma 3. *Let $\delta(t) = \max_{a \leq \alpha \leq b} |F(\alpha, t) - 1|$. Then*

$$\begin{aligned} \int_0^\infty G(u, t) \min \left\{ \left(\frac{u}{x} \right)^{2\beta}, \left(\frac{x}{u} \right)^{2\beta} \right\} \frac{du}{u} &= \frac{t}{2\pi\beta} \log t + R, \\ R &\ll \frac{t(\log t)}{\beta} \left\{ \delta(t) + \left(\frac{x}{t^b} \right)^{2\beta} \log t + \left(\frac{t^a}{x} \right)^{2\beta} \log t \right\}, \end{aligned} \tag{2}$$

uniformly for $\beta \geq 1$ and $t^a \leq x \leq t^b$.

We substitute $u = t^\alpha$ in the left hand side of (2). Since $G(t^\alpha, t) = (2\pi)^{-1}t(\log t)F(\alpha, t)$, this yields

$$\frac{t}{2\pi}(\log t)^2 \int_{-\infty}^{\infty} F(\alpha, t) \min(x^{-2\beta}t^{2\alpha\beta}, x^{2\beta}t^{-2\alpha\beta})d\alpha.$$

On using the trivial estimate $F(\alpha, t) \ll \log t$ for the ranges $\alpha < a$ and $\alpha > b$, we see that the above expression is

$$\begin{aligned} & \frac{t}{2\pi}(\log t)^2 \int_a^b F(\alpha, t) \min(x^{-2\beta}t^{2\alpha\beta}, x^{2\beta}t^{-2\alpha\beta})d\alpha \\ & + O\left(\frac{t(\log t)^2}{\beta} \left\{ \left(\frac{x}{t^b}\right)^{2\beta} + \left(\frac{t^a}{x}\right)^{2\beta} \right\}\right). \end{aligned}$$

By the definition of $\delta(t)$ this is

$$\frac{t}{2\pi\beta}(\log t) + O\left(\frac{t\delta(t)(\log t)}{\beta}\right) + O\left(\frac{t(\log t)^2}{\beta} \left\{ \left(\frac{x}{t^b}\right)^{2\beta} + \left(\frac{t^a}{x}\right)^{2\beta} \right\}\right),$$

as required.

Lemma 4. Let $\varepsilon > 0$, and define $\eta(T)$ to be the supremum of $\delta(t)$ for $T^{1/2} \leq t \leq T$. If $\eta(T) \leq 1$ then

$$S(T, x) \ll_\varepsilon T(1 + \eta(T)^{1/4}(\log T)^{1/2}),$$

uniformly for $T^{a+\varepsilon} \leq x \leq T^{b-\varepsilon}$.

Let $\beta \geq 1$. From Lemmas 2 and 3 we have

$$\begin{aligned} G_\beta(x, t) &= \frac{t}{2\pi}(\log t) + O(\beta^3 R) \\ &= \frac{t}{2\pi}(\log t) + O(\beta^2 t \delta(t)(\log t)) + O(\beta^2 t^{1-\varepsilon/2}), \end{aligned}$$

uniformly for $t^{a+\varepsilon/2} \leq x \leq t^{b-\varepsilon/2}$. It follows that

$$\max_{T(\log T)^{-2} \leq t \leq T} G_\beta(x, t) \ll_\varepsilon T(\log T) + \beta^2 T \eta(T)(\log T) + \beta^2 T^{1-\varepsilon/2},$$

uniformly for $T^{a+\varepsilon} \leq x \leq T^{b-\varepsilon}$. For $t \leq T(\log T)^{-2}$ we use the trivial estimate $G_\beta(x, t) \ll \beta t(\log t)^2$, whence

$$\max_{t \leq T(\log T)^{-2}} G_\beta(x, t) \ll \beta T.$$

Lemma 1 now yields

$$S(T, x) \ll_\varepsilon T + T\beta^{-1/2}(\log T)^{1/2} + (\beta\eta(T))^{1/2}T(\log T)^{1/2} + \beta^{1/2}T^{1-\varepsilon/4}.$$

On choosing $\beta = \min(\eta(T)^{-1/2}, \log T)$ we obtain Lemma 4.

We now examine (1). Let $p_n^{1/2} \leq \Delta \leq p_n^{1/2} \log p_n$ and $a + \varepsilon < 2 < b - \varepsilon$. Then if $p_n/\Delta \leq T \ll p_n^{3/2}(\log p_n)\Delta^{-2}$ and $p_n \leq x \leq p_{n+1}$ we will have $T^{a+\varepsilon} \leq x \leq T^{b-\varepsilon}$ whenever p_n is large enough. From (1) and Lemma 4 we therefore obtain

$$1 + \eta(T)^{1/4}(\log p_n)^{1/2} \gg \Delta^2 T p_n^{-3/2} \gg \Delta p_n^{-1/2}.$$

However the hypothesis of our theorem is that $\eta(T) \rightarrow 0$ as $T \rightarrow \infty$, and so $A = o(p_n^{1/2}(\log p_n)^{1/2})$ as claimed.

References

1. Cramér, H.: Some theorems concerning prime numbers. *Ark. Mat. Astronom. Fys.* **15**, 1–32 (1920)
2. Heath-Brown, D.R.: Gaps between primes, and the pair correlation of zeros of the zeta-function. *Acta Arith.* **41**, 85–99 (1982)
3. Montgomery, H.L.: *Topics in multiplicative number theory*. Berlin, Heidelberg, New York: Springer 1971
4. Montgomery, H.L.: The pair correlation of zeros of the zeta-function. *Proc. Symp. Pure Math.* **24**, 181–193 (1973)
5. Mueller, J.H.: On the difference between consecutive primes, *Recent progress in analytic number theory, I*, pp. 269–273. London, New York: Academic Press 1981

Received March 30, 1983