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A Note on the Differences Between Consecutive Primes

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1. Introduction

The distribution of primes has recently been shown to be related to the pair correlation of the zeros of the Riemann Zeta-function $\zeta(s)$. The correlation is measured by the function

$$F(\alpha, T)=\left(\frac{T}{2\pi}\log T\right)^{-1}\sum_{0<\gamma,\gamma'\leqq T} T^{i\alpha(\gamma-\gamma')}\frac{4}{4+(\gamma-\gamma')^2},$$

where α and $T\geqq 2$ are real and γ, γ' run over the imaginary parts of the zeros of $\zeta(s)$. On the Riemann Hypothesis, Montgomery [4] proved that

$$F(\alpha, T)=(1+o(1))T^{-2\alpha}\log T+\alpha+o(1)$$

as $T\to\infty$, uniformly for $0\leqq\alpha\leqq 1-\varepsilon$, if $\varepsilon>0$. Moreover he conjectured that $F(\alpha, T)\to 1$ as $T\to\infty$, uniformly for $1\leqq\alpha\leqq M$, for any fixed $M>1$. This would have important implications, both for the zeta-function and for the distribution of primes. In particular Heath-Brown [2], improving a result of Mueller [5], showed that Montgomery's conjecture, together with the Riemann Hypothesis, leads to

$$p_{n+1}-p_n=O(p_n^{1/2}(\log p_n)^{1/2}).$$

Here, only the fact that $F(\alpha, T)\ll 1$ was used. The above bound may be compared with the classical result of Cramér [1], that

$$p_{n+1}-p_n=O(p_n^{1/2}(\log p_n))$$

on the Riemann Hypothesis. The purpose of this note is to show how the full strength of Montgomery's conjecture can be applied to this problem.

Theorem. *Assume the Riemann Hypothesis. Suppose further that there exist a, b with $1<a<2<b$ such that $F(\alpha, T)\to 1$ as $T\to\infty$, uniformly for $a\leqq\alpha\leqq b$. Then*

$$p_{n+1}-p_n=o(p_n^{1/2}(\log p_n)^{1/2}).$$

2. Proof of the Theorem

We assume the Riemann Hypothesis only for (1) below. It is not needed elsewhere. We first convert the problem into one depending on the distribution of the numbers γ. Write $p_{n+1}-p_n=\Delta$ and

$$S(T,x)=\left|\sum_{0<\gamma\leqq T} x^{i\gamma}\right|.$$

By the method of Mueller [5; Sect. 2] we find that there is a numerical constant C_0 such that if $\Delta\geqq p_n^{1/2}$, then

$$S(T,x)\geqq C_0\Delta^2T^2p_n^{-3/2} \tag{1}$$

for some T, x satisfying $T\geqq p_n/\Delta$, $p_n\leqq x\leqq p_{n+1}$. Since $S(T,x)\ll T\log T$ it is clear that (1) is false unless $T\ll p_n^{3/2}(\log p_n)\Delta^{-2}$, as we henceforth assume.

Our first lemma is a generalization of Heath-Brown [2; Lemma 4]. Let

$$\begin{aligned}G_\beta(x,T)&=\sum_{0<\gamma,\gamma'\leqq T} x^{i(\gamma-\gamma')}\frac{4\beta^2}{4\beta^2+(\gamma-\gamma')^2}\\&=\beta\int_{-\infty}^{\infty} e^{-2\beta|u|}\left|\sum_{0<\gamma\leqq T} x^{i\gamma}e^{i\gamma u}\right|^2du.\end{aligned}$$

Lemma 1. *We have* $S(T,x)\ll(T\beta^{-1}\max_{t\leqq T} G_\beta(x,t))^{1/2}$ *uniformly for* $0<\beta\leqq T$ *and* $x\geqq 1$.

The proof is the same as in [2; Sect. 2] except that the extra parameter β is introduced by applying the Sobolev-Gallagher inequality [3; Lemma 1.1] in the form

$$|f(0)|\leqq\frac{1}{2\delta}\int_{-\delta}^{\delta}|f(u)|du+\tfrac{1}{2}\int_{-\delta}^{\delta}|f'(u)|du,$$

with $\delta=1/\beta$.

Lemma 2. *Write* $G_1(x,t)=G(x,t)$. *For* $x>0$, $\beta>0$, $t\geqq 2$, *we have*

$$G_\beta(x,t)=\beta^2G(x,t)+\beta(1-\beta^2)\int_0^\infty G(u,t)\min\left\{\left(\frac{u}{x}\right)^{2\beta},\left(\frac{x}{u}\right)^{2\beta}\right\}\frac{du}{u}.$$

This follows immediately, on noting that

$$\int_0^\infty u^{iv}\min\left\{\left(\frac{u}{x}\right)^{2\beta},\left(\frac{x}{u}\right)^{2\beta}\right\}\frac{du}{u}=\frac{4\beta x^{iv}}{4\beta^2+v^2}.$$

Lemma 3. *Let* $\delta(t)=\max_{a\leqq\alpha\leqq b}|F(\alpha,t)-1|$. *Then*

$$\begin{aligned}&\int_0^\infty G(u,t)\min\left\{\left(\frac{u}{x}\right)^{2\beta},\left(\frac{x}{u}\right)^{2\beta}\right\}\frac{du}{u}=\frac{t}{2\pi\beta}\log t+R,\\&R\ll\frac{t(\log t)}{\beta}\left\{\delta(t)+\left(\frac{x}{t^b}\right)^{2\beta}\log t+\left(\frac{t^a}{x}\right)^{2\beta}\log t\right\},\end{aligned} \tag{2}$$

uniformly for $\beta\geqq 1$ *and* $t^a\leqq x\leqq t^b$.

We substitute $u=t^\alpha$ in the left hand side of (2). Since $G(t^\alpha, t) =(2\pi)^{-1}t(\log t)F(\alpha, t)$, this yields

$$\frac{t}{2\pi}(\log t)^2 \int_{-\infty}^{\infty} F(\alpha, t) \min(x^{-2\beta}t^{2\alpha\beta}, x^{2\beta}t^{-2\alpha\beta})d\alpha .$$

On using the trivial estimate $F(\alpha, t) \ll \log t$ for the ranges $\alpha < a$ and $\alpha > b$, we see that the above expression is

$$\frac{t}{2\pi}(\log t)^2 \int_{a}^{b} F(\alpha, t) \min(x^{-2\beta}t^{2\alpha\beta}, x^{2\beta}t^{-2\alpha\beta})d\alpha$$
$$+O\left(\frac{t(\log t)^2}{\beta}\left\{\left(\frac{x}{t^b}\right)^{2\beta}+\left(\frac{t^a}{x}\right)^{2\beta}\right\}\right).$$

By the definition of $\delta(t)$ this is

$$\frac{t}{2\pi\beta}(\log t)+O\left(\frac{t\delta(t)(\log t)}{\beta}\right)+O\left(\frac{t(\log t)^2}{\beta}\left\{\left(\frac{x}{t^b}\right)^{2\beta}+\left(\frac{t^a}{x}\right)^{2\beta}\right\}\right),$$

as required.

Lemma 4. *Let $\varepsilon>0$, and define $\eta(T)$ to be the supremum of $\delta(t)$ for $T^{1/2}\leqq t\leqq T$. If $\eta(T)\leqq 1$ then*

$$S(T, x) \ll_\varepsilon T(1+\eta(T)^{1/4}(\log T)^{1/2}),$$

uniformly for $T^{a+\varepsilon}\leqq x\leqq T^{b-\varepsilon}$.

Let $\beta\geqq 1$. From Lemmas 2 and 3 we have

$$G_\beta(x, t)=\frac{t}{2\pi}(\log t)+O(\beta^3 R)$$
$$=\frac{t}{2\pi}(\log t)+O(\beta^2 t\delta(t)(\log t))+O(\beta^2 t^{1-\varepsilon/2}),$$

uniformly for $t^{a+\varepsilon/2}\leqq x\leqq t^{b-\varepsilon/2}$. It follows that

$$\max_{T(\log T)^{-2}\leqq t\leqq T} G_\beta(x, t)\ll_\varepsilon T(\log T)+\beta^2 T\eta(T)(\log T)+\beta^2 T^{1-\varepsilon/2},$$

uniformly for $T^{a+\varepsilon}\leqq x\leqq T^{b-\varepsilon}$. For $t\leqq T(\log T)^{-2}$ we use the trivial estimate $G_\beta(x, t)\ll \beta t(\log t)^2$, whence

$$\max_{t\leqq T(\log T)^{-2}} G_\beta(x, t)\ll \beta T.$$

Lemma 1 now yields

$$S(T, x)\ll_\varepsilon T+T\beta^{-1/2}(\log T)^{1/2}+(\beta\eta(T))^{1/2}T(\log T)^{1/2}+\beta^{1/2}T^{1-\varepsilon/4}.$$

On choosing $\beta=\min(\eta(T)^{-1/2}, \log T)$ we obtain Lemma 4.

We now examine (1). Let $p_n^{1/2}\leqq \Delta\leqq p_n^{1/2}\log p_n$, and $a+\varepsilon<2<b-\varepsilon$. Then if $p_n/\Delta\leqq T\ll p_n^{3/2}(\log p_n)\Delta^{-2}$ and $p_n\leqq x\leqq p_{n+1}$ we will have $T^{a+\varepsilon}\leqq x\leqq T^{b-\varepsilon}$ whenever p_n is large enough. From (1) and Lemma 4 we therefore obtain

$$1+\eta(T)^{1/4}(\log p_n)^{1/2}\gg \Delta^2 Tp_n^{-3/2}\gg \Delta p_n^{-1/2}.$$

However the hypothesis of our theorem is that $\eta(T) \to 0$ as $T \to \infty$, and so $\Delta = o(p_n^{1/2}(\log p_n)^{1/2})$ as claimed.

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