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SIEVING THE POSITIVE INTEGERS
BY SMALL PRIMES

D. A. GOLDSTON AND KEVIN S. McCURLEY

ABSTRACT. Let $Q$ be a set of primes that has relative density $\delta$ among the primes, and let $\phi(x, y, Q)$ be the number of positive integers $\leq x$ that have no prime factor $\leq y$ from the set $Q$. Standard sieve methods do not seem to give an asymptotic formula for $\phi(x, y, Q)$ in the case that $\frac{1}{2} \leq \delta < 1$. We use a method of Hildebrand to prove that

$$\phi(x, y, Q) \sim xf(u) \prod_{p \leq y} \left(1 - \frac{1}{p}\right)$$

as $x \to \infty$, where $u = \frac{\log x}{\log y}$ and $f(u)$ is defined by

$$u^\delta f(u) = \begin{cases} \frac{\gamma^{\gamma \delta}}{\Gamma(1-\delta)}, & 0 < u \leq 1, \\ \frac{\gamma^{\gamma \delta}}{\Gamma(1-\delta)} + \delta \int_0^u f(t)(1+t)^{\delta-1} \, dt, & u > 1. \end{cases}$$

This may also be viewed as a generalization of work by Buchstab and de Bruijn, who considered the case where $Q$ consisted of all primes.

1. Introduction. Let $Q$ be a set of primes, and let $\phi(x, y, Q)$ be the number of positive integers $\leq x$ that have no prime factor $\leq y$ from the set $Q$. In the case that $Q$ consists of all primes, it was proved by Buchstab [2] that

$$\phi(x, y, Q) \sim xe^{\gamma}(u) \prod_{p \leq y} \left(1 - \frac{1}{p}\right),$$

where $u = \frac{\log x}{\log y}$ is fixed and $\omega(\cdot)$ is defined by

$$u \omega(u) = \begin{cases} 0, & 0 < u \leq 1, \\ 1 + \int_0^{u-1} \omega(t) \, dt, & u > 1. \end{cases}$$

Uniform estimates for this case were later given by de Bruijn [1].

Our aim in the present paper is to give an asymptotic estimate for $\phi(x, y, Q)$ in the case that the set $Q$ has relative density $\delta$ among the primes, where $0 < \delta < 1$. If $Q$ is a set of primes, we define

$$\psi(x, Q) = \sum \log p,$$

where $p \leq x$ and $p \in Q$. 

Received by the editors August 13, 1987 and, in revised form, September 25, 1987.


The research of the first author was supported by a grant from the National Science Foundation.

The research of the second author was supported by a grant from the National Security Agency.
For a set $Q$ of primes, we use $Q^c$ to denote the set of primes that are not in $Q$. Throughout the rest of this paper we shall assume that $Q$ has relative density $\delta$ among the primes, or more precisely that if $R(x)$ is defined by $\psi(x, Q^c) = (1 - \delta)x + xR(x)$, then there exists a constant $A > 1$ such that

$$R(x) = O\left(\frac{1}{\log^A x}\right)$$

For example, it follows from the prime number theorem for arithmetic progressions that if $Q$ consists of the primes belonging to a finite union of arithmetic progressions, then (1) is satisfied for every $A > 0$.

In the case that (1) is satisfied with $0 < \delta < 1$, estimates for $\phi(x, y, Q)$ can be obtained by a variety of sieve methods, and in particular it follows from a "fundamental lemma" (see [4, Theorem 2.5]) that

$$\phi(x, y, Q) = xW(y)\left[1 + O\left(\exp\left(-\sqrt{\log x}\right)\right) + O\left(\exp\left(-u(\log u - \log \log u - \log \delta - 2)\right)\right)\right],$$

where throughout this paper we shall write $u = \frac{\log x}{\log y}$ and

$$W(y) = \prod_{p \leq y} \left(1 - \frac{1}{p}\right).$$

If $u \to \infty$, then this gives an asymptotic estimate for $\phi(x, y, Q)$. When $u$ is constant (2) no longer gives an asymptotic for $\phi(x, y, Q)$, but in the case that $Q$ satisfies (1) with $0 < \delta < \frac{1}{2}$, Iwaniec [7] used the sieve of Eratosthenes-Legendre to prove a general sieve result that gives

$$\phi(x, y, Q) = xW(y)\left(f(u) + O\left(\frac{u + 1}{(1 - 2\delta)(\log y)^{1 - 2\delta}}\right)\right),$$

where $f(\cdot)$ is defined by

$$u^\delta f(u) = \begin{cases} \frac{e^{\gamma \delta}}{\Gamma(1 - \delta)}, & 0 < u \leq 1, \\ \frac{e^{\gamma \delta}}{\Gamma(1 - \delta)} + \delta \int_0^{u-1} f(t)(1 + t)^{\delta - 1} dt, & u > 1. \end{cases}$$

(Note the similarity to the Buchstab function when $\delta = 1$.)

For the case when $\frac{1}{2} \leq \delta < 1$, the Rosser Sieve [8] gives the best known result, namely that

$$g_1(u)xW(y)(1 + o(1)) \leq \phi(x, y, Q) \leq g_2(u)xW(y)(1 + o(1)),$$

where $g_1$ and $g_2$ are continuous solutions of

$$u^\delta g_2(u) = A, \quad u \leq \beta + 1,$$

$$u^\delta g_1(u) = B, \quad u \leq \beta,$$

$$(u^\delta g_2(u))' = \delta u^{\delta - 1} g_1(u - 1), \quad u > \beta + 1,$$

$$(u^\delta g_1(u))' = \delta u^{\delta - 1} g_2(u - 1), \quad u > \beta,$$

and $A$, $B$, and $\beta$ are constants that depend on $\delta$. One might expect that an asymptotic relation similar to (3) holds as well in the case that $\frac{1}{2} \leq \delta < 1$, and this is precisely what we shall prove. Our main result is the following.
THEOREM. Let \( Q \) be a set of primes satisfying (1) with \( 0 < \delta < 1 \) and \( A > 1 \). Then for any \( \gamma \) with \( 0 < \gamma < 1 - \delta \) there exists a constant \( C \) such that

\[
\phi(x, y, Q) = x f(u) W(y) \left( 1 + O \left( \frac{1}{\log \log y} + \frac{\log(1 + u)}{\log^2 y} \right) \right)
\]

uniformly for \( x \geq y \geq \exp(C(\log \log x)^{1/\gamma}) \), where \( u = \frac{\log x}{\log y} \) and \( f \) is defined by (4).

We have made little effort to optimize the bound for the error term and the range of uniformity for \( y \), and some improvements are possible (we will say more on this later). The primary importance of our theorem is for fixed \( u \), since (2) already gives an asymptotic relation for \( \phi(x, y, Q) \) in the case that \( u \to \infty \). Note however that the bound on the relative error in our asymptotic expression is smaller than that of (2) in the case that \( u \) tends slowly to infinity.

Our method of proof for the theorem is similar to that used by Hildebrand [5], and was also used by the authors in a previous paper [3] where we treated the case of sieving by the primes greater than \( y \) that are in \( Q \). The method is based on the “Buchstab-type” identity

\[
\phi(x, y, Q) \log x = \int_1^x \frac{\phi(t, y, Q)}{t} dt + \sum_{\substack{p^n \leq x \\ \ \ p \leq y \\ \ \ p \notin Q}} \phi \left( \frac{x}{p^m}, y, Q \right) \log p
\]

\[
+ \sum_{\substack{p^n \leq x \\ \ \ y < p \leq x \\ \ \ p \notin Q}} \phi \left( \frac{x}{p^m}, y, Q \right) \log p.
\]

(Throughout this paper the letter \( p \) denotes a prime number.)

Most sieve methods (see [4]) would make use of a Buchstab-type identity to estimate the sifting function \( \phi(x, x^{1/u}, Q) \) in terms of sifting functions \( \phi(x, x^{1/u_1}, Q) \) with \( u_1 > u \). The method of Hildebrand that we use in this paper works in the opposite direction, using estimates for smaller values of \( u \) rather than larger values. In both cases we eventually arrive at a range of values of \( u \) where it is possible to estimate the sifting function directly. In the case of \( \phi(x, y, Q) \) when \( Q \) consists of all primes, this is easy since we have \( \phi(x, y, Q) = \pi(x) - \pi(y) \) if \( 1 \leq u \leq 2 \). In the case that \( Q \) has density less than one, the situation is somewhat more complicated, and in place of the prime number theorem we shall appeal to a result of Wirsing.

Specifically, it follows from [10, Satz 3] that if \( Q \) satisfies (1) with \( 0 < \delta < 1 \), then

\[
\phi(x, x, Q) = \frac{e^{(\delta - 1) \gamma} x}{\Gamma(1 - \delta)} \log x \prod_{\substack{p \leq x \\ \ \ p \notin Q}} \left( 1 - \frac{1}{p} \right)^{-1} \left( 1 + O \left( \frac{1}{\log \log 3x} \right) \right).
\]

In fact, [10] gives (6) from a much weaker hypothesis than (1). From (1) it may be proved in a standard manner (see [6, pp. 22 24]) that

\[
W(y) = \frac{B}{\log \delta} \left( 1 + O \left( \frac{1}{\log y} \right) \right)
\]
for some constant $B$. Hence if $x < y$, then
\[
\frac{W(y)}{W(x)} = \prod_{\substack{x < p \leq y \\ p \in Q}} \left( 1 - \frac{1}{p} \right) = \left( \frac{\log x}{\log y} \right)^{\delta} \left( 1 + O \left( \frac{1}{\log x} \right) \right).
\]
Combining this with the result
\[
\prod_{p \leq x} \left( 1 - \frac{1}{p} \right) = \frac{e^{-\gamma}}{\log x} \left( 1 + O \left( \frac{1}{\log x} \right) \right),
\]
(see [6, p. 24]) it follows from (6) that
\[
(7) \quad \phi(x, y, Q) = x W(y) f(u) \left( 1 + O \left( \frac{1}{\log \log 3x} \right) \right).
\]
if $0 < u \leq 1$.

It may be possible to improve on (6), and any improvement of the error term in (6) will result in an improvement of the error term in our main theorem when $u$ is small. In particular if $Q$ consists of the primes $\equiv 3 \pmod{4}$, then the error term $O\left( \frac{1}{\log x} \right)$ can probably be improved to $O\left( \frac{1}{\log^{2} x} \right)$, since one can apply essentially the same analytic argument as Landau (see [9, pp. 257–263]) for estimating the number of integers $\leq x$ that can be represented as the sum of two squares. Another improvement can be realized by applying the two step reduction process used in [3] to obtain a better bound for the error term when $u$ is large (or equivalently, when $y$ is small compared to $x$). This can be used to produce better short interval results for $\phi(x, y, Q)$.

2. Preliminary lemmas. Let $0 < \delta < 1$. We define the functions $f(u) = f_\delta(u)$ by the equation (4). These functions were studied previously by Iwaniec [7] (with slightly different notation). The properties that we require are summarized in the following lemma.

LEMMA 1. If $0 < \delta < 1$, then the function $f(u)$ satisfies
\begin{enumerate}
\item[(i)] $u f(u) = (1 - \delta) \int_{u-1}^{u} f(t) \, dt + \int_{0}^{u-1} f(t) \, dt$, $u \geq 1$,
\item[(ii)] $f(u) = 1 + O(e^{-u})$, $u \geq 1$,
\item[(iii)] $f'(u) = O(e^{-u})$, $u \geq 1$.
\end{enumerate}

PROOF. If we differentiate both sides of (4), then we find that $uf''(u) = \delta f(u-1) - \delta f(u)$, for $u > 1$. Now define
\[
g(u) = u f(u) - (1 - \delta) \int_{u-1}^{u} f(t) \, dt - \int_{0}^{u-1} f(t) \, dt.
\]
Differentiating, we find that $g'(u) = 0$ for $u > 1$. Hence $g$ is constant, and by letting $u$ tend to 1, we see that the constant is zero. (ii) and (iii) were proved by Iwaniec [7].
Lemma 2. If \( y \leq x \), then

\[
\sum_{\substack{y < p^m \leq x \atop m \geq 2}} \phi \left( \frac{x}{p^m}, y, Q \right) \log p \ll xy^{-1/2}.
\]

Proof. Let \( \psi^*(t) = \sum_{m \geq 2} p^m \log p \). It follows from the prime number theorem that \( \psi^*(t) \ll t^{1/2} \). Hence from the trivial estimate \( \phi(t, y, Q) \leq t \) we obtain

\[
\sum_{\substack{y < p^m \leq x \atop m \geq 2}} \phi \left( \frac{x}{p^m}, y, Q \right) \log p \leq x \sum_{\substack{y < p^m \leq x \atop m \geq 2}} \frac{\log p}{p^m}
= x \int_y^x \frac{d\psi^*(t)}{t}
= \psi^*(x) - x \frac{\psi^*(y)}{y} + x \int_y^x \frac{\psi^*(t)}{t^2} dt
\ll x^{1/2} + x \int_y^x t^{-3/2} dt,
\]

which proves the result. □

Lemma 3. If \( 0 < \varepsilon < 0.5 \) and \( u \geq 1 + \varepsilon \), then

\[
\sum_{y^u < p \leq y^u} \phi \left( \frac{y^u}{p}, y, Q \right) \log p \ll \frac{y^u}{\log y^u} + \varepsilon y^u \log y.
\]

Proof. Let \( \theta(t) = \sum_{p \leq t} \log p \). It follows from the prime number theorem that \( \theta(t) = t + O(\log^{-1} t) \). Hence

\[
\sum_{y^u < p \leq y^u} \phi \left( \frac{y^u}{p}, y, Q \right) \log p \leq y^u \sum_{y^u < p \leq y^u} \frac{\log p}{p}
= y^u \int_{y^u - \varepsilon}^{y^u} \frac{d\theta(t)}{t}
= y^u r(y^u) - y^u r(y^u - \varepsilon) + y^u \int_{y^u - \varepsilon}^{y^u} \frac{\theta(t) dt}{t^2}
\ll \frac{y^u}{\log y^u} + \varepsilon y^u \log y. \quad \Box
\]

Lemma 4. If \( 0 < \alpha \leq 1, u \geq 1 + \varepsilon \), and \( 0 < \varepsilon < 0.5 \), then

\[
\sum_{p^m \leq y^u} \frac{\log p}{p^m} f \left( u - \frac{\log p^m}{\log y} \right) = (1 - \delta) \log y \int_{u-\alpha}^{u} f(t) dt
+ O(1 + \varepsilon^{-\delta} \log^{-A} y),
\]

where the implied \( O \) constant may depend on \( \alpha \).
PROOF. We integrate by parts and replace $(t, QC)$ by $(1-\delta)t + tR(t)$ to obtain

$$\sum_{p^m \leq y^\alpha} \frac{\log p}{p^m} f\left( u - \frac{\log p^m}{\log y} \right) = \int_{y^\alpha}^{y^\alpha} f\left( u - \frac{\log t}{\log y} \right) dv(t, QC)$$

$$= \psi(y^\alpha, QC)f(u - \alpha)$$

$$+ \int_{u-\alpha}^{u} \{ f(w) \log y + f'(w) \} y^{w-u} \psi(y^{u-w}, QC) \, dw$$

$$= T_1 + T_2,$$

where

$$T_1 = (1-\delta)f(u - \alpha) + (1-\delta) \int_{u-\alpha}^{u} \{ f(w) \log y + f'(w) \} \, dw$$

$$= (1-\delta) \log y \int_{u-\alpha}^{u} f(w) \, dw + O(1),$$

and

$$T_2 = f(u - \alpha)R(y^\alpha) + \int_{u-\alpha}^{u} \{ f(w) \log y + f'(w) \} \frac{R(y^{u-w})}{(u-w)^A} \, dw$$

$$\ll \epsilon^{-\delta} \log^{-A} y + \log^{1-A} y \int_{u-\alpha}^{u} f(w) \, dw \frac{R(y^{u-w})}{(u-w)^A} \, dw$$

$$+ \log^{-A} y \int_{\epsilon}^{u} \frac{|f'(w)|}{(u-w)^A} \, dw.$$

Let $u_1 = u - \frac{\log 2}{\log y} - \frac{\alpha}{2}$, the midpoint of the interval of integration. Then

$$\int_{u-\alpha}^{u} \frac{f(w) \, dw}{(u-w)^A} \ll (u - u_1)^{-A} \int_{u-\alpha}^{u_1} f(w) \, dw + \int_{u_1}^{u} \frac{dw}{(u-w)^A} \ll 1 + \log^{A-1} y.$$

We will now consider several cases for estimating the second integral of (8). If $u \leq 1 + \frac{\log 2}{\log y}$, then we have

$$\int_{\epsilon}^{u} \frac{|f'(w)|}{(u-w)^A} \, dw \ll \int_{\epsilon}^{1/2} w^{-\delta-1} \, dw + \int_{1/2}^{u} (u-w)^{-A} \, dw$$

$$\ll \epsilon^{-\delta} + \log^{A-1} y.$$
The first of these integrals can be treated as in the previous case, and the second satisfies

\[ \int_1^{u - \frac{\log 2}{\log y}} \frac{|f'(w)|}{(u - w)^A} dw \ll \log^A y \int_1^2 |f'(w)| dw \ll \log^A y. \]

If finally we have \( u - \frac{\log 2}{\log y} > 2 \), then we can estimate the contribution from \( \varepsilon < w < 1.5 \) as before, and by Lemma 1 the remainder is dominated by

\[ \int_{1.5}^{u - \frac{\log 2}{\log y}} \frac{e^{-w}}{(u - w)^A} dw \ll \log^A y. \]

This completes the proof.

**Lemma 5.** If \( 0 < \varepsilon < 0.5 \) and \( u \geq 1 + \varepsilon \), then

\[ \sum_{y < p \leq y^{u - \varepsilon}} \frac{\log p}{p} f \left( u - \frac{\log p}{\log y} \right) \]

\[ = \log y \int_0^{u-1} f(t) dt + O(\varepsilon^{1-\delta} \log y + \varepsilon^{-\delta} \log^{-A} y + \log^{1-A} y). \]

**Proof.** With \( \theta(t) \) and \( r(t) \) as in the proof of Lemma 3, integration by parts yields

\[ \sum_{y < p \leq y^{u - \varepsilon}} \frac{\log p}{p} f \left( u - \frac{\log p}{\log y} \right) = \int_y^{y^{u - \varepsilon}} \frac{f(u - \frac{\log t}{\log y})}{t} d\theta(t) = T_1 + T_2, \]

where

\[ T_1 = f(\varepsilon) - f(u - 1) + \int_y^{y^{u - \varepsilon}} \left\{ f \left( u - \frac{\log t}{\log y} \right) + \frac{f'(u - \frac{\log t}{\log y})}{\log y} \right\} \frac{dt}{t} \]

\[ = \log y \int_\varepsilon^{u-1} f(w) dw, \]

and

\[ T_2 = r(y^{u - \varepsilon})f(\varepsilon) - r(y)f(u - 1) \]

\[ + \int_y^{y^{u - \varepsilon}} \left\{ f \left( u - \frac{\log t}{\log y} \right) + \frac{f'(u - \frac{\log t}{\log y})}{\log y} \right\} \frac{r(t) dt}{t}. \]

Note that

\[ T_1 - \log y \int_0^{u-1} f(w) dw = - \log y \int_0^{\varepsilon} f(w) dw \]

\[ = O(\varepsilon^{1-\delta} \log y). \]

The prime number theorem implies that \( r(t) \ll \log^{-A} t \), so that

\[ T_2 \ll \varepsilon^{-\delta} \log^{-A} y + \log^{1-A} y \int_0^{u-1} \frac{f(w)}{(u - w)^A} dw \]

\[ + \log^{-A} y \int_\varepsilon^{u-1} \frac{|f'(w)|}{(u - w)^A} dw. \]
If \( u \geq 2 \), then the first integral satisfies
\[
\int_0^{u-1} \frac{f(w)}{(u-w)^A} dw \ll \int_0^1 f(w) \, dw + \int_1^{u-1} \frac{dw}{(u-w)^A} \ll 1.
\]

If \( u < 2 \), then the first integral of (9) is bounded by \( \int_0^1 f(w) \, dw \ll 1 \).

For \( u \geq 3 \), we use Lemma 1 to estimate the second integral of (9) as
\[
\int_\varepsilon^{u-1} \frac{|f'(w)|}{(u-w)^A} dw \ll \int_\varepsilon^{2} |f'(w)| \, dw + \int_2^{u-1} e^{-w} \, dw \ll \varepsilon^{-\delta} + 1.
\]

If \( 1 + \varepsilon \leq u \leq 3 \), then this last estimate may be proved in essentially the same manner, and combining these estimates proves the lemma.

**3. Proof of the theorem.** In order to now prove the theorem, we first give a proof of (5). Let \( P(y) \) denote the product of the primes belonging to \( Q \) that do not exceed \( y \). Then
\[
\sum_{n \leq x, (n, P(y)) = 1} \log n = \sum_{n \leq x} \sum_{p^m | n} \log p
\]
\[
= \sum_{p^m \leq x} \sum_{n \leq x, p \not| P(y)} \log p \cdot 1
\]
\[
= \sum_{p^m \leq x} \phi \left( \frac{x}{p^m}, y, Q \right) \log p
\]
\[
+ \sum_{p^m \leq x} \phi \left( \frac{x}{p^m}, y, Q \right) \log p.
\]

On the other hand we can integrate by parts to get
\[
\sum_{n \leq x, (n, P(y)) = 1} \log n = \int_1^x \log t \, d\phi(t, y, Q)
\]
\[
= \phi(x, y, Q) \log x - \int_1^x \frac{\phi(t, y, Q)}{t} \, dt,
\]
and this proves (5).

For \( u > 0 \) and \( y \geq 1.5 \), we define \( \Delta(u, y) \) by the relation
\[
\phi(yu, y, Q) = y^u W(y) f(u)(1 + \Delta(u, y)).
\]
Furthermore for \( 0 < \varepsilon < 0.5 \) and \( u \geq \varepsilon \), we define
\[
\Delta^*(u, y) = \sup_{\varepsilon \leq u' \leq u} |\Delta(u', y)|
\]
\[
\Delta^{**}(u, y) = \sup_{0 \leq u' \leq u} |\Delta(u', y)|.
\]
The initial conditions (7) imply that
\[ \Delta (u, y) \ll \frac{1}{\log \log (3y^u)} \]
uniformly for \(0 < u \leq 1\). Let \(\gamma\) be fixed, with \(0 < \gamma < 1 - \delta\), and define \(\varepsilon\) by
\[ \varepsilon = \log^{-\delta-\gamma} y. \]

In order to prove the theorem it suffices to prove that
\[ \Delta^*(u, y) \ll \frac{1}{\log \log 3y} + \frac{\log(1+u)}{\log^\gamma y}, \]
uniformly for \(y \geq 1.5\) and \(u \geq \frac{1}{2}\). Without loss of generality we may assume that \(y\) is sufficiently large. It follows from (10) that if \(0.5 \leq u \leq 1\), then (11) is satisfied. For \(u > 1\), we will now use the identity (5) to estimate \(\Delta^*(u, y)\) in terms of \(\Delta^*(u', y)\) with some \(\frac{1}{2} \leq u' \leq 1\).

Let
\[ m = \inf_{t \geq 1/2} f(t), \quad M = \sup_{t \geq 1/2} f(t). \]

It follows from Lemma 1 that \(0 < m < M < \infty\). For \(u \geq 1\), we now define
\[ \lambda = \min \left\{ \frac{1}{2}, \frac{m}{2(1-\delta)M} \right\}, \]
\[ \alpha_1 = \frac{1-\delta}{uf(u)} \int_{u-\lambda}^{u} f(t) \, dt, \]
\[ \alpha_2 = \frac{1-\delta}{uf(u)} \int_{u-1}^{u-\lambda} f(t) \, dt, \]
\[ \alpha_3 = \frac{1}{uf(u)} \int_{0}^{u-1} f(t) \, dt. \]

It follows from Lemma 1 that \(\alpha_1 + \alpha_2 + \alpha_3 = 1\), and clearly by the choice of \(\lambda\) we have \(\alpha_1 \leq \frac{1}{2}\).

Let \(u \geq 1 + \varepsilon\). We begin by rewriting (5) in terms of \(\Delta\) to obtain
\[ 1 + \Delta (u, y) = R_1 + R_2 \]
\[ + \frac{1}{f(u) \log y^u} \sum_{p^m \leq y} \frac{\log p}{p^m} f \left( u - \frac{\log p^m}{\log y} \right) \left( 1 + \Delta \left( u - \frac{\log p^m}{\log y}, y \right) \right), \]
\[ + \frac{1}{f(u) \log y^u} \sum_{y < p \leq y^{u-\epsilon}} \frac{\log p}{p} f \left( u - \frac{\log p}{\log y} \right) \left( 1 + \Delta \left( u - \frac{\log p}{\log y}, y \right) \right), \]
\[ + \frac{1}{f(u) y^u \log y^u} \int_1^{y^u} f \left( \frac{\log t}{\log y} \right) \left( 1 + \Delta \left( \frac{\log t}{\log y}, y \right) \right) \, dt, \]
where
\[ R_1 = \frac{1}{W(y) f(u) y^u \log y^u} \sum_{y < p^m \leq y^u \atop p \leq y \Rightarrow p \not\in Q \atop m \geq 2} \phi \left( \frac{y^u}{p^m}, y, Q \right) \log p, \]
\[ R_2 = \frac{1}{W(y) f(u) y^u \log y^u} \sum_{y^{u-\epsilon} < p \leq y^u} \phi \left( \frac{y^u}{p}, y, Q \right) \log p. \]
From this we obtain for \( u \geq 1 + \varepsilon \) that

\[
|\Delta(u, y)| \leq |R_1| + |R_2| + (1 + \Delta^{**}(u, y))|R_3|
\]

\[+ |R_4| + |R_5| + |R_6| \]

\[+ \frac{\Delta^*(u, y)}{f(u) \log y^u} \sum_{p \leq y^u} \frac{\log p}{p^m} f\left(u - \frac{\log p^m}{\log y}\right) \]

\[+ \frac{\Delta^*(u - \lambda, y)}{f(u) \log y^u} \sum_{y^u < p < y^u} \frac{\log p}{p^m} f\left(u - \frac{\log p^m}{\log y}\right) \]

\[+ \frac{\Delta^*(u - 1, y)}{f(u) \log y^u} \sum_{y^u < p \leq y^u - \varepsilon} \frac{\log p}{p} f\left(u - \frac{\log p}{\log y}\right) \]

\[\leq \alpha_1 \Delta^*(u, y) + \alpha_2 \Delta^*(u - \lambda, y) + \alpha_3 \Delta^*(u - 1, y) \]

\[+ O\left((1 + \Delta^{**}(u, y)) \sum_{i=1}^{6} |R_i|\right), \]

where

\[R_3 = \frac{1}{f(u) y^u \log y^u} \int_{1}^{y^u} f\left(\frac{\log t}{\log y}\right) \, dt,\]

\[R_4 = \frac{1}{f(u) \log y^u} \sum_{p \leq y^u} \frac{\log p}{p^m} f\left(u - \frac{\log p^m}{\log y}\right) - \alpha_1,\]

\[R_5 = \frac{1}{f(u) \log y^u} \sum_{y^u < p \leq y^u - \varepsilon} \frac{\log p}{p^m} f\left(u - \frac{\log p^m}{\log y}\right) - \alpha_2,\]

\[R_6 = \frac{1}{f(u) \log y^u} \sum_{y^u < p \leq y^u - \varepsilon} \frac{\log p}{p} f\left(u - \frac{\log p}{\log y}\right) - \alpha_3.\]

The term \( R_1 \) is \( \ll (u \log y)^{-1} \) by Lemma 2, and Lemma 3 yields

\[|R_2| \ll (u \log y)^{-1} + \varepsilon u^{-1} \log^\delta y \ll u^{-1} \log^{-\gamma} y.\]

The term \( R_3 \) satisfies

\[|R_3| = \frac{1}{f(u) y^u} \int_{0}^{u} y^w f(w) \, dw \ll \frac{1}{y^{u \varepsilon}} \left(y^{1/2} \int_{0}^{1/2} w^{-\delta} \, dw + \int_{1/2}^{u} y^w \, dw\right) \ll (u \log y)^{-1}.\]

Lemma 4 yields the estimate

\[|R_4| + |R_5| \ll \frac{1 + \varepsilon^{-\delta} \log^{-\Lambda} y}{u \log y} \ll (u \log y)^{-1},\]
and Lemma 5 yields the estimate

\[ |R_6| \ll \left( \frac{\varepsilon^{1-\delta} \log y + \varepsilon^{-\delta} \log^{-A} y + \log^{1-A} y}{u \log y} \right) \]
\[ \ll u^{-1} \log^{-\gamma} y. \]

From (10) we may deduce that \( \Delta^{**}(u, y) \leq C + \Delta^*(u, y) \) for \( u \geq \varepsilon \) and some \( C > 0 \). Combining these estimates with (13) and the fact that \( \Delta^* \) is nondecreasing gives

\[ |\Delta(u, y)| \leq \alpha_1 \Delta^*(u, y) + (\alpha_2 + \alpha_3) \Delta^*(u - \lambda, y) \]
\[ + O\left( \frac{1 + \Delta^*(u, y)}{u \log^7 y} \right). \]

Since \( \alpha_1 \leq 0.5 \) we have
\[ 0.5(\Delta^*(u, y) + \Delta^*(u - \lambda, y)) - \alpha_1 \Delta^*(u, y) - (\alpha_2 + \alpha_3) \Delta^*(u - \lambda, y) \]
\[ = (0.5 - \alpha_1)(\Delta^*(u, y) - \Delta^*(u - \lambda, y)), \]
and the latter quantity is clearly nonnegative. Hence (14) gives
\[ |\Delta(u, y)| \leq 0.5 \Delta^*(u, y) + 0.5 \Delta^*(u - \lambda, y) + O\left( \frac{1 + \Delta^*(u, y)}{u \log^7 y} \right). \]

The same argument as in [5, pp. 300–301] gives

\[ \Delta^*(u, y) \leq \Delta^*(u - \lambda, y) + O\left( \frac{1 + \Delta^*(u, y)}{u \log^7 y} \right), \]

uniformly for \( u \geq 1 + \varepsilon \). We iterate this bound to obtain

\[ \Delta^*(u, y) \leq \Delta^*(u_0, y) + O\left( \frac{(1 + \Delta^*(u, y)) \log(1 + u)}{\log^7 y} \right) \]

for some \( u_0 \) with \( 1 + \varepsilon - \lambda \leq u_0 < 1 + \varepsilon \). From (15) we now obtain
\[ \Delta^*(u_0, y) \leq \Delta^*(1 + \varepsilon, y) \leq \Delta^*(1 + \varepsilon - \lambda, y) + O\left( \frac{1 + \Delta^*(1 + \varepsilon, y)}{\log^7 y} \right). \]

Since \( 1 + \varepsilon - \lambda < 1 \) for \( y \) sufficiently large, this gives
\[ \Delta^*(u_0, y) \ll \frac{1}{\log \log y} + \frac{1 + \Delta^*(u, y)}{\log^7 y}. \]

The result now follows from (16), provided \( \frac{\log(1+u)}{\log^7 y} \) is sufficiently small. The latter condition is satisfied if the constant \( C \) of the theorem is chosen sufficiently large.

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