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A NOTE ON $S(t)$ AND THE ZEROS OF THE RIEMANN ZETA-FUNCTION

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Abstract

Let $\pi S(t)$ denote the argument of the Riemann zeta-function at the point $1/2+it$. Assuming the Riemann hypothesis, we sharpen the constant in the best currently known bounds for $S(t)$ and for the change of $S(t)$ in intervals. We then deduce estimates for the largest multiplicity of a zero of the zeta-function, and for the largest gap between the zeros.

1. *Introduction*

We assume the Riemann hypothesis (RH) throughout this paper.

Let $N(t)$ denote the number of zeros $\rho = 1/2 + i\gamma$ of the Riemann zeta-function with ordinates in the interval $(0, t]$. Then, for $t \geqslant 2$,

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\Big(\frac{1}{t}\Big), \tag{1.1}$$

where, if t is not the ordinate of a zero, $S(t)$ denotes the value of $(1/\pi)\arg\zeta(1/2+it)$ obtained by continuous variation along the straight line segments joining 2, $2+it$, and $1/2+it$, starting with the value 0 (see [**10**]). If t is the ordinate of a zero, we set

$$S(t) = \frac{1}{2}\lim_{\varepsilon\to 0^+}\{S(t+\varepsilon) + S(t-\varepsilon)\}.$$

It follows from (1.1) that

$$N(t+h) - N(t) = \frac{h}{2\pi}\log\frac{t}{2\pi} + S(t+h) - S(t) + O\bigg(\frac{(1+h^2)}{t}\bigg) \tag{1.2}$$

for $0 < h \leqslant t$. Littlewood [**5**] proved, assuming the Riemann hypothesis, that

$$S(t) \ll \frac{\log t}{\log\log t}, \tag{1.3}$$

where here the notation $f \ll g$ means the same as $f = O(g)$. Hence the number of zeros with ordinates in an interval $(t, t+h]$ satisfies

$$N(t+h) - N(t) - \frac{h}{2\pi}\log\frac{t}{2\pi} \ll \frac{\log t}{\log\log t}, \tag{1.4}$$

provided that $0 < h \leqslant \sqrt{t}$, say.

The bounds in (1.3) and (1.4) have not been improved over the last eighty years, except in the size of the implied constants. Our goal in this paper is to sharpen these results.

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THEOREM 1. *Assume the Riemann hypothesis. Let t be large, and let $0 < h \leqslant \sqrt{t}$. Then we have*

$$\Bigl|N(t+h) - N(t) - \frac{h}{2\pi}\log\frac{t}{2\pi}\Bigr| \leqslant \Bigl(\frac{1}{2} + o(1)\Bigr)\frac{\log t}{\log\log t}. \tag{1.5}$$

In light of (1.2), this is equivalent to saying that

$$\Bigl|S(t+h) - S(t)\Bigr| \leqslant \Bigl(\frac{1}{2} + o(1)\Bigr)\frac{\log t}{\log\log t} \tag{1.6}$$

for $0 < h \leqslant \sqrt{t}$. Using this, we obtain the following theorem.

THEOREM 2. *Assume the Riemann hypothesis. Then for t sufficiently large we have*

$$|S(t)| \leqslant \Bigl(\frac{1}{2} + o(1)\Bigr)\frac{\log t}{\log\log t}. \tag{1.7}$$

To deduce Theorem 2 from Theorem 1, we use the (unconditional) estimate of Littlewood [5], that

$$\int_0^T S(u)\,du \ll \log T,$$

which implies that

$$\int_t^{t+\log^2 t} S(u)\,du \ll \log t\,.$$

Therefore, for t sufficiently large, there is an h with $0 \leqslant h \leqslant \log^2 t$ such that $S(t+h) \leqslant 1$. Rewriting (1.6) as

$$-\Bigl(\frac{1}{2} + o(1)\Bigr)\frac{\log t}{\log\log t} + S(t+h) \leqslant S(t) \leqslant \Bigl(\frac{1}{2} + o(1)\Bigr)\frac{\log t}{\log\log t} + S(t+h), \tag{1.8}$$

we obtain the upper bound for $S(t)$ from the right-hand inequality. We obtain the lower bound by using an h for which $S(t+h) \geqslant -1$, together with the left-hand inequality.

The following is an almost immediate corollary of Theorem 1.

COROLLARY 1. *Assume the Riemann hypothesis. Let $m(\gamma)$ denote the multiplicity of the zero $1/2 + i\gamma$. Then if γ is sufficiently large, we have*

$$m(\gamma) \leqslant \Bigl(\frac{1}{2} + o(1)\Bigr)\frac{\log \gamma}{\log\log \gamma}. \tag{1.9}$$

Moreover, if γ and γ' are consecutive ordinates and $\gamma < \gamma'$, then

$$\gamma' - \gamma \leqslant \frac{\pi}{\log\log\gamma}(1 + o(1)). \tag{1.10}$$

To deduce (1.9), take $t = \gamma - h/2$ in (1.5) with $h = o(1/\log\log\gamma)$. To deduce (1.10), assume that $N(t+h) - N(t) = 0$ in (1.5), and solve for h.

There has been some earlier work on Theorem 2. In place of the constant 1/2, Ramachandra and Sankaranarayanan [8] obtained 1.119 . . . in 1993, and Fujii [2] obtained .67 in 2004. For a recent survey of this area, see [4]. It is interesting that Brumer [1] obtained the same constant 1/2 for a similar bound for the rank of an elliptic curve in terms of the conductor.

2. *Proof of Theorem* 1

We begin by stating two lemmas. The first is a form of the Guinand–Weil explicit formula.

LEMMA 1. *Let $h(s)$ be analytic in the strip $|\operatorname{Im} s| \leqslant 1/2 + \varepsilon$ for some $\varepsilon > 0$, and assume that $|h(s)| \ll (1 + |s|)^{-(1+\delta)}$ for some $\delta > 0$ when $|\operatorname{Re} s| \to \infty$. Let $h(w)$ be real-valued for real w, and set $\hat{h}(x) = \int_{-\infty}^{\infty} h(w) e^{-2\pi i x w}\, dw$. Then*

$$\sum_\rho h\Big(\frac{\rho - 1/2}{i}\Big) = h\Big(\frac{1}{2i}\Big) + h\Big(-\frac{1}{2i}\Big) - \frac{1}{2\pi}\hat{h}(0)\log\pi + \frac{1}{2\pi}\int_{-\infty}^{\infty} h(u)\ \operatorname{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\, du \\ - \frac{1}{2\pi}\sum_{n=2}^{\infty} \frac{\Lambda(n)}{\sqrt{n}}\Big(\hat{h}\Big(\frac{\log n}{2\pi}\Big) + \hat{h}\Big(\frac{-\log n}{2\pi}\Big)\Big), \tag{2.1}$$

where Γ'/Γ is the logarithmic derivative of the gamma function, and $\Lambda(n)$ is the von Mangoldt function defined to be $\log p$ if $n = p^m$, p a prime and $m \geqslant 1$, and zero otherwise.

This is a specialization of [3, Theorem 5.12], and in particular [3, equation (25.10)]. The conditions in [3] are that $\hat{h}$ is an infinitely differentiable function with compact support, which will be satisfied in our application below; however, it is also not hard to prove the lemma with the conditions that we have stated.

LEMMA 2. *Let L and δ be positive real numbers, and let $w = u + iv$. There exist even entire functions $F_+(w)$ and $F_-(w)$ with the following properties:*

(i) *$F_-(u) \leqslant \chi_{[-L,L]}(u) \leqslant F_+(u)$ for all real u;*
(ii) *$\int_{-\infty}^{\infty} F_+(u)\, du \leqslant 2L + 1/\delta$, and $\int_{-\infty}^{\infty} F_-(u)\, du \geqslant 2L - 1/\delta$;*
(iii) *$F_\pm(w) \ll e^{2\pi\delta|\operatorname{Im} w|}$;*
(iv) *$F_\pm(u) \ll \min(1,\, \delta^{-2}(|u| - L)^{-2})$ for $|u| > L$;*
(v) *$\hat{F}_\pm(x) = 0$ for $|x| \geqslant \delta$;*
(vi) *$\hat{F}_\pm(x) = (\sin 2\pi L x)/\pi x + O(1/\delta)$.*

This is essentially [7, Lemma 2]. Functions of this type were constructed by A. Selberg, who gives a nice discussion of them in [9]. For a proof of this lemma, see H. L. Montgomery [6] and J. D. Vaaler [11]. The slightly less familiar property (iv) is obtained from [11, Lemma 5].

To prove Theorem 1, we use Lemma 1 with $h(w) = F(w - t)$, where t is large and positive and F denotes either the function F_+ or F_- from Lemma 2. We assume that the parameters δ and L implicit in the definition of $F_\pm$ satisfy the conditions

$$\delta \geqslant 1 \qquad \text{and} \qquad 0 < L \leqslant 2\sqrt{t}. \tag{2.2}$$

Clearly, $\hat{h}(x) = e^{-2\pi i x t}\hat{F}(x)$. Therefore, by Lemma 2(i) and Lemma 2(ii), or Lemma 2(vi), we have

$$\hat{h}(0) = \hat{F}(0) = 2L + O\Big(\frac{1}{\delta}\Big),$$

and

$$\hat{h}\Big(\frac{\log n}{2\pi}\Big) = n^{-it}\hat{F}\Big(\frac{\log n}{2\pi}\Big), \qquad \hat{h}\Big(-\frac{\log n}{2\pi}\Big) = n^{it}\hat{F}\Big(-\frac{\log n}{2\pi}\Big).$$

We also see that

$$h\Big(\frac{1}{2i}\Big) + h\Big(-\frac{1}{2i}\Big) = F\Big(\frac{1}{2i} - t\Big) + F\Big(\frac{1}{2i} - t\Big) \ll e^{\pi\delta}$$

by Lemma 2(iii).

We will now show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(u - t)\operatorname{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{iu}{2}\Big)\, du = \frac{1}{2\pi}\Big(\log\frac{t}{2}\Big)\hat{F}(0) + O(1). \tag{2.3}$$

First, since

$$\operatorname{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{iu}{2}\Big) \ll \log(|u|+2),$$

we see by Lemma 2(iv) and (2.2) that

$$\begin{aligned}\int_{t+4\sqrt{t}}^{\infty} F(u-t)\,\operatorname{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{iu}{2}\Big)\,du &\ll \int_{t+4\sqrt{t}}^{\infty} \frac{\log(u+2)}{\delta^2(u-t-2\sqrt{t})^2}\,du \\ &\ll \frac{\log t}{\sqrt{t}},\end{aligned}$$

and similarly for the integral over $(-\infty, t-4\sqrt{t}]$. Next, by Stirling's formula for large t, together with Lemma 2(ii) and the previous argument using Lemma 2(iv), we have

$$\begin{aligned}\int_{t-4\sqrt{t}}^{t+4\sqrt{t}} F(u-t)\,\operatorname{Re}\frac{\Gamma'}{\Gamma}(\frac{1}{4}+\frac{iu}{2})\,du &= \int_{t-4\sqrt{t}}^{t+4\sqrt{t}} F(u-t)\left(\log\frac{u}{2}+O\left(\frac{1}{1+u^2}\right)\right)du \\ &= \int_{t-4\sqrt{t}}^{t+4\sqrt{t}} F(u-t)\left(\log\frac{t}{2}+O\left(\frac{1}{\sqrt{t}}\right)\right)du \\ &= \Big(\log\frac{t}{2}\Big)\int_{-\infty}^{\infty} F(u-t)\,du + O\left(\frac{\log(t/2)+L}{\sqrt{t}}\right) \\ &= \Big(\log\frac{t}{2}\Big)\hat{F}(0) + O\left(1\right).\end{aligned}$$

On combining these estimates, (2.3) follows.

Inserting these results into (2.1), we obtain

$$\sum_{\gamma} F(\gamma-t) = \frac{\hat{F}(0)}{2\pi}\log\Big(\frac{t}{2\pi}\Big) + O(e^{\pi\delta}) - \frac{1}{2\pi}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}\left(n^{-it}\hat{F}\Big(\frac{\log n}{2\pi}\Big)+n^{it}\hat{F}\Big(\frac{-\log n}{2\pi}\Big)\right). \tag{2.4}$$

By Lemma 2(v) and Lemma 2(vi), the sum on the right is

$$\ll \sum_{n<e^{2\pi\delta}} \frac{\Lambda(n)}{\sqrt{n}}\cos(t\log n)\left(\frac{\sin(L\log n)}{\log n}+O\Big(\frac{1}{\delta}\Big)\right) \ll \sum_{n\leqslant e^{2\pi\delta}}\frac{1}{\sqrt{n}} \ll e^{\pi\delta},$$

where the last sum was estimated trivially. Hence

$$\sum_{\gamma} F(\gamma-t) = \frac{\hat{F}(0)}{2\pi}\log\Big(\frac{t}{2\pi}\Big) + O(e^{\pi\delta}). \tag{2.5}$$

Taking F to be F_+ and using Lemma 2(i) and Lemma 2(ii), we find that

$$N(t+L)-N(t-L) \leqslant \frac{1}{2\pi}\log\Big(\frac{t}{2\pi}\Big)\Big(2L+\frac{1}{\delta}\Big) + O(e^{\pi\delta})\,.$$

We now take $\pi\delta = \log\log t - 2\log\log\log t$ and obtain

$$N(t+L)-N(t-L) - \frac{L}{\pi}\log\Big(\frac{t}{2\pi}\Big) \leqslant \Big(\frac{1}{2}+o(1)\Big)\frac{\log t}{\log\log t}\,.$$

Had we used F_- in (2.5) instead of F_+, we would have found that

$$N(t+L)-N(t-L) - \frac{L}{\pi}\log\Big(\frac{t}{2\pi}\Big) \geqslant \Big(-\frac{1}{2}+o(1)\Big)\frac{\log t}{\log\log t}.$$

Combining these two inequalities, we conclude that

$$\left|N(t+L)-N(t-L) - \frac{L}{\pi}\log\Big(\frac{t}{2\pi}\Big)\right| \leqslant \Big(\frac{1}{2}+o(1)\Big)\frac{\log t}{\log\log t}\,.$$

Finally, replacing t by $t+h/2$ and taking $L=h/2$, we obtain Theorem 1.

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