EXPLAINING THE DECIMAL EXPANSION OF 1/81 USING CALCULUS

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In the course of doing a numerical calculation, we needed the decimal value of 1/81. The answer according to a HP15C calculator was

$$\frac{1}{81} = 0.012345679.$$

We naturally concluded that the obvious pattern should be correct, and that actually 1/81 = 0.0123456789..., only the calculator had rounded the 89 to a 9. However, on going to a computer we found

$$\frac{1}{81} = 0.012345679012345679012345679012345679\ldots,$$

so our conclusion was wrong.

The explanation of this phenomena is not hard to guess: in our base ten decimal system the number 9 has the important property that 9 = 10 − 1, or equivalently $10 \equiv 1 \pmod 9$. Thus

$$\frac{1}{81} = \frac{1}{9^2} = \frac{1}{(10-1)^2} = \frac{1}{10^2\left(1-\frac{1}{10}\right)^2}.$$

The geometric series is given by

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \ldots, \quad \text{for } |x|<1,$$

and on differentiating both sides (valid by a standard calculus theorem) we obtain

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots, \quad \text{for } |x|<1.$$

Letting $x = 1/10$ in the above we obtain

$$\frac{1}{81} = \frac{1}{10^2}\left(1 + \frac{2}{10} + \frac{3}{10^2} + \frac{4}{10^3} + \frac{5}{10^4} + \ldots .\right)$$

$$= 0\times10^{-1} + 1\times10^{-2} + 2\times10^{-3} + 3\times10^{-4} + 4\times10^{-5} + 5\times10^{-6} + \ldots$$

$$\begin{array}{ll} = & \begin{array}{r} 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,2\,2\,2 \\ 0\,.\,0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,0\,1\,2 \end{array} \cdots \end{array}$$

$$\begin{array}{ll} =+ & \begin{array}{r} 1\,1\,1\,1\,1\,1\,1\,1\,1\,1\,2\,2\,2\,2 \\ \underline{0\,.\,0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,0\,1\,2\,3\,4\,5\,6\,7\,8\,9\,0\,1\,2} \end{array} \cdots \end{array}$$

$$= \quad 0\,.\,0\,1\,2\,3\,4\,5\,6\,7\,9\,0\,1\,2\,3\,4\,5\,6\,7\,9\,0\,1\,2\,3\,4 \ldots .$$

We thus see that our initial (wrong) guess was actually correct: the decimal expansion of 1/81 really is the sequence of non-negative integers, only this fact is disguised by the overlapping digits having been added. Thus, for example, the first missing 8 in the decimal expansion has been turned into a 9 after we add to the 8 the carry digit from adding 9 to the overlapping 1 in the 10.

Not just 1/81 leads to this phenomena:

$$1/891 = 0.001122334455667789001122\ldots,$$

$$1/8991 = 0.000111222333444555666777889000111222\ldots,$$

and so on. These fractions give the means to remove the carry digits which obscure the pattern in 1/81, as may be observed by considering

$$\overline{.0123456789} = \frac{1}{81} - \frac{10}{89999999991}.$$

The reader interested in more facts about decimal expansions of small fractions should consult [1]. Further properties of the number 9 may be found in [2].

REFERENCES

1 O. D. Anderson, "Repeating Decimals," *Mathematics and Computer Education,* Vol. 24, No. 2, 135-147 (Spring 1990).

2 A. H. Beiler, *Recreations in the Theory of Numbers,* 2nd Edition, Dover Publications (1966).