Answers for some of the homework problems from sections 4.1-7.5

4.1.14 If \( n = 1 \), then there is only one string. If \( n \geq 2 \) there are \( 2^{n-2} \) bit strings with the desired property.

4.1.20.
(a) Every ninth number is divisible by 9, so the answer is \( \frac{9000}{9} = 1000 \).
(b) Every other number is even, so the answer is \( \frac{9000}{2} = 4500 \).
(c) We can reason in various ways, but the shortest solution is probably the following. From left to right: there are 9 choices for the first digit (it can not be zero), then 9 choices for the second digit (it can’t equal the first digit), then in a similar way, 8 choices for the third, and 7 choices for the third. Therefore there are \( 9 \cdot 9 \cdot 8 \cdot 7 = 4536 \) ways to specify such a number.
(d) Every third number is divisible by 3, so the answer is \( 9000 - \frac{9000}{3} = 9000 - 3000 = 6000 \).
(e) There are 1800 of the numbers divisible by 5 and 1286 of them divisible by 7. There are 257 numbers which are divisible by 35 (i.e. by both 5 and 7) and therefore the answer is \( 1800 + 1286 - 257 = 2829 \).
(f) We use part (e) and find that the answer is \( 9000 - 2829 = 6171 \).
(g) We use part (e) and find that the answer is \( 1800 - 257 = 1543 \).
(h) 257.

4.1.22.
(a) Similarly to 4.1.20(c) we find that the answer is \( 10 \cdot 9 \cdot 8 \cdot 7 = 5040 \).
(b) \( 10^3 \cdot 5 = 5000 \) (or the half of all strings).
(c) There are four ways to choose the position and nine ways to chose the digit, thus the answer is \( 4 \cdot 9 = 36 \).

4.2.6. There are only \( d \) possible remainders (objects) when an integer is divided by \( d \), namely 0, 1, ..., \( d - 1 \). By the pigeonhole principle (at least) two out of the \( d + 1 \) integers (boxes) must have the same remainders.

4.2.16. We apply the pigeonhole principle for the four pairs

\[ \{1, 15\}, \{3, 13\}, \{5, 11\}, \{7, 9\} \]

4.2.30. There are 99,999,999 possible salaries (boxes) less than one million dollars, i.e., from $0.01 to $999,999.99. By the pigeonhole principle at least two of the earners have the same salary.

4.3.10. \( P(6, 6) = 6! = 720 \).

4.3.12.
(a) \( C(12, 3) = 220 \)
(b) \( C(12, 3) + C(12, 2) + C(12, 1) + C(12, 0) = 220 + 66 + 12 + 1 = 299 \)
(c) \( 2^{12} - C(12, 2) - C(12, 1) - C(12, 0) = 4017 \)
(d) \( C(12, 6) = 924 \)
4.3.22.  
(a) If we think for $ED$ as a superletter, then we count permutations of seven superletters: $A, B, C, F, G, H, ED$. Thus the answer is $P(7, 7) = 7! = 5040$.
(b) Reasoning as in (a) we find that the answer is $P(6, 6) = 6! = 720$.
(c) $P(5, 5) = 5! = 120$.
(d) $P(5, 5) = 5! = 120$.
(e) $P(4, 4) = 4! = 24$.
(f) There are no such permutations since $B$ cannot be followed by both $C$ and $F$ at the same time.

4.4.6. $C(13, 8) = 1287$.

4.4.8. $C(17, 9)3^82^9 = 81, 662, 929, 920$.

6.1.4. (Most of them done in class!)
(a) $-3a_{n-1} + 4a_{n-2} = -3 \cdot 0 + 4 \cdot 0 = 0 = a_n$.
(b) $a_n = a_{n-1} = a_{n-2} = ... = (-1)^k a_{n-k} = ... = (-1)^n a_0$
Therefore $a_n = 5 \cdot (-1)^n$.
(c) $a_n = 3 + a_{n-1} = 3 + 3 + a_{n-2} = ... = k \cdot 3 + a_{n-k} = ... = n \cdot 3 + a_0$.
Thus $a_n = 3n + 1$.

(d) Similarly to (c) we find:
\[
a_n = -3 + 2a_{n-1} \\
= -3 - 2 \cdot 3 + 2^2 a_{n-2} \\
\vdots \\
= -3 - 3 \cdot 2 - \ldots - 3 \cdot 2^{k-1} + 2^k a_{n-k} \\
\vdots \\
= -(3 + 3 \cdot 2 + \ldots + 3 \cdot 2^{n-1}) + 2^n a_0
\]
Similarly to (c) and (e) we find:

\[ a_n = (n + 1)a_{n-1} = (n + 1)na_{n-2} \]
\[ \vdots \]
\[ = (n + 1)n...(n - (k - 2))a_{n-k} \]
\[ \vdots \]
\[ = (n + 1)n...2a_0 \]
\[ = 2(n + 1)! \]

(f)

\[ a_n = 2na_{n-1} \]
\[ \vdots \]
\[ = 2^k n(n - 1)...(n - (k - 1))a_{n-k} \]
\[ \vdots \]
\[ = 2^n n(n - 1)...1a_0 \]
\[ = 3 \cdot 2^n n! \]

(g) (Solved in class) Since \( a_n = n - 1 - a_{n-1} = 1 + a_{n-2} \) we easily find: \( a_n = k + a_0 \) if \( n = 2k \) and \( a_n = k + a_1 \) if \( n = 2k + 1 \). Thus \( a_n = k + 7 \) for \( n = 2k \) and \( a_n = k - 7 \) for \( n = 2k + 1 \).

6.2.4.

(d) The characteristic polynomial is \( r^2 = 2r - 1 \) or \( r^2 - 2r + 1 = 0 \). The zeros are \( r = 1, 1 \) (double root 1) and therefore \( a_n = \alpha_1 1^n + \alpha_2 n1^n = \alpha_1 + n\alpha_2 \). Furthermore, we find \( 4 = a_0 = \alpha_1 \) and \( 1 = a_1 = \alpha_1 + \alpha_2 \). Thus \( \alpha_2 = -3 \) and \( a_n = 4 - 3n \).

(e) Similarly to part (d) we find that \( r^2 - 1 = 0 \) has roots \( r = 1, -1 \). Thus \( a_n = \alpha_1 (-1)^n + \alpha_2 \). On the other hand \( 5 = a_0 = \alpha_1 + \alpha_2 \) and \( -1 = a_2 = -\alpha_1 + \alpha_2 \). Therefore \( \alpha_1 = 3 \), \( \alpha_2 = 2 \), and \( a_n = 3 \cdot (-1)^n + 2 \).

(f) The roots are \( r = -3, -3 \), i.e. \( a_n = \alpha_1 (-3)^n + \alpha_2 n(-3)^n \). We easily find \( \alpha_1 = 3 \) and \( \alpha_2 = -2 \) and thus \( a_n = (3 - 2n)(-3)^n \).

(g) The roots are \( r = -5, 1 \), i.e. \( a_n = \alpha_1 (-5)^n + \alpha_2 \). We easily find \( \alpha_1 = -1 \) and \( \alpha_2 = 3 \) and thus \( a_n = (-5)^n + 3 \).

6.2.28. (Solved in class)

(a) The associated homogeneous recurrence relation is \( a_n = 2a_{n-1} \) which has solution \( a_n^{(h)} = \alpha 2^n \). We seek for a particular solution of the form \( a_n = p_2 n^2 + p_1 n + p_0 \) and after plugging in the recurrence relation we find

\[ p_2 n^2 + p_1 n + p_0 = 2(p_2 (n - 1)^2 + p_1 (n - 1) + p_0) + 2n^2 \]
After we solve this polynomial equation (a system of three equations with three unknowns $p_0, p_1, p_2$) we find $p_0 = -12$, $p_1 = -8$, and $p_2 = -2$. Therefore $a_n = \alpha 2^n - 2n^2 - 8n - 12$.

(b) From $4 = a_1 = 2\alpha - 2 - 8 - 12$ we find $\alpha = 13$ and $a_n = 13 \cdot 2^n - 2n^2 - 8n - 12$.

7.1.6.
(a) The relation is:
- **not** reflexive ($1 + 1 \neq 0$);
- symmetric ($x + y = y + x$);
- **not** antisymmetric ($(1, -1)$ and $(-1, 1)$ are in $R$ but $1 \neq -1$);
- **not** transitive ($(1, -1) \in R$ and $(-1, 1) \in R$ but $(1, 1) \notin R$).

(b) Similarly to (a) we find that the relation is:
- reflexive;
- symmetric ($x = \pm y$ implies $y = \pm x$);
- **not** antisymmetric ($(1, -1)$ and $(-1, 1)$ are in $R$ but $1 \neq -1$);
- transitive.

(c) Similarly to (a) and (b) we find that the relation is:
- reflexive;
- symmetric ($x - y = \frac{\xi}{\eta}$ implies $y - x = -\frac{\xi}{\eta}$);
- **not** antisymmetric ($(1, 0)$ and $(0, 1)$ are in $R$ but $1 \neq 0$);
- transitive (sum of two rational numbers is rational).

(c) Similarly to (a), (b), and (c), we find that the relation is:
- **not** reflexive ($1 \neq 2 \cdot 1$);
- **not** symmetric ($(1, 2) \in R$ but $(2, 1) \notin R$);
- antisymmetric ($x = 2y$ and $y = 2x$ imply $x = y = 0$);
- **not** transitive ($(4, 2) \in R$, $(2, 1) \in R$, but $(4, 1) \notin R$).

7.1.32.
(a) The union of the relations is $R_1 \cup R_3$. Here this means that the first number is greater than the second or vice versa, in other words, the two numbers are not equal. This is just the relation $R_6$.

(b) We must have $a > b$ or $a = b$, i.e. $a \geq b$. The answer is $R_2$.

(c) The relation $R_2 \cap R_4$ happens if $a \geq b$ and $a \leq b$, i.e. precisely when $a = b$. The answer is $R_5$.

7.1.34.
(a) For $(a, c)$ to be in $R_1 \circ R_1$, we must find an element $b$ such that $a > b$ and $b > c$. This can be done is and only if $a > c$, i.e. $R_1 \circ R_1 = R_1$.

(b) As in (a) we see that $R_1 \circ R_2 = R_1$.

(c) For $(a, c)$ to be in $R_1 \circ R_3$, we must find an element $b$ such that $a < b$ and $b > c$. This can be done by choosing $b$ to be large enough for any $a$ and $c$. Thus the answer is $R_1 \circ R_3 = R^2$.

(d) As in (c) we see that $R_1 \circ R_4 = R^2$.

7.5.10. This follows from 7.5.5 (done in class) for $f(a, b) := \frac{a}{b}$. The function from the set of pairs of positive integers to the set of positive rational numbers. We have that $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. 

Only parts (a) and (d) are equivalence relations. In part (a) there is one equivalence class for each \( n \in \mathbb{Z} \), and it contains all those functions whose value at 1 is \( n \). In part (d) the set of equivalence classes is too large (it is uncountable).