FINAL EXAM – 80 points

1 (20 pts) Let $F$ be a field of characteristic $p$, and $a$ be an element of $F$ not of the form $b^p - b$, $b \in F$. Determine the Galois group over $F$ of a splitting field of $x^p - x - a$. 
2 (20 pts) Assume that $x^p - x$, $a \in \mathbb{Q}$, is irreducible in $\mathbb{Q}[x]$. Show that the Galois group of $x^p - a$ over $\mathbb{Q}$ is isomorphic to the group of transformations of $\mathbb{Z}_p$ of the form $y \mapsto ky + l$, where $k, l \in \mathbb{Z}_p$ and $k \neq 0$. 
3 (20 pts) If $L = \mathfrak{sl}(n, F)$ and $g \in GL(n, F)$ (the group of $n \times n$ invertible matrices with entries in $F$). Prove that the map $\delta : L \to L$ defined by $\delta : x \mapsto -gx^t g^{-1}$ ($x^t$ is the transpose of $x$) is an automorphism of $L$. Prove that if $n = 2$ and $g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $\delta$ is inner, i.e. there is an element $y$ in $\mathfrak{sl}(2, F)$ for which $\exp(\text{ad}(y))(x) = \delta(x) = -x^t$ for every $x$ in $\mathfrak{sl}(2, F)$. 
4 (20 pts) Prove that $L$ is solvable Lie algebra if and only if there exists a chain of subalgebras

$$L = L_0 \supset L_1 \supset ... \supset L_k = 0$$

such that $L_{i+1}$ is an ideal of $L_i$ and such that each quotient $L_i/L_{i+1}$ is abelian.