A bijection on core partitions

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My research relates to the study of symmetry, with applications to Chemistry, Physics, Differential Equations . . . .

In this talk we’ll describe some geometry associated to the symmetric group of permutations.
Symmetric Group

The symmetric group $S_n$ is the set of all bijections

$$\sigma : \{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots n\}$$

with composition as the group operation.

For example, $\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{bmatrix}$ and $\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix}$ are permutations in $S_5$ and if we compose them, we get

$$\tau \sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{bmatrix}.$$
Symmetric Group

The *symmetric group* $S_n$ has a presentation with generators

$$s_1, s_2, \ldots, s_{n-1}$$

and relations

$$s_i^2 = id$$

$$s_is_j = s_js_i \quad \text{for } |i - j| \geq 2,$$

$$s_is_i+1s_i = s_{i+1}s_is_{i+1}$$

We think of each $s_i$ as an *adjacent transposition*:

$$s_i = \begin{bmatrix} 1 & 2 & \cdots & i & i+1 & \cdots & n \\ 1 & 2 & \cdots & i+1 & i & \cdots & n \end{bmatrix}.$$
Symmetric Group

Groups with a similar presentation in terms of generators and relations are called Coxeter groups. They have generators $s_1, s_2, \ldots, s_n$ with each $s_i^2 = id$, and

$$(s_is_j)^{m_{ij}} = id.$$ 

for some $m_{ij} \geq 2$. The relation $s_i^2 = id$ means that $s_i = s_i^{-1}$.

For example,

$id = (s_is_j)^2 = s_is_js_is_j$ 

is equivalent to

$s_js_i = s_is_j$ 

and

$id = (s_is_j)^3 = s_is_js_is_js_is_j$ 

is equivalent to

$s_js_is_j = s_is_js_i$
The relations in a Coxeter group are often visualized in a combinatorial graph.

- Vertices = generators.
- No edge $\iff (s_i s_j)^2 = id \iff s_i s_j = s_j s_i$.
- Unlabeled edge $\iff (s_i s_j)^3 = id \iff s_i s_j s_i = s_j s_i s_j$.
- Edge labeled by $m$ $\iff (s_i s_j)^m = id \iff s_i s_j s_i \cdots = s_j s_i s_j \cdots$.

For example, $S_n$ has the Coxeter graph

```
●s_1    ●s_2    ●s_3  ...  ●s_{n-2}    ●s_{n-1}
```
The relation $s_i^2 = id$ means that $s_i = s_i^{-1}$. We can view the generators $s_i$ as reflections of a vector space.

**Definition**

Let $u \in \mathbb{R}^n$. A *reflection* through the hyperplane orthogonal to $u$ is a linear map $s_u$ sending

$$v \mapsto v - \frac{2 \langle u, v \rangle}{\langle u, u \rangle} u$$
Symmetric Group

For $S_n$, we can combine our two points of view if we take a vector space $\mathbb{R}^n$ with orthonormal basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$.

E.g. let $n = 3$. Then define $\alpha_1 = \varepsilon_1 - \varepsilon_2$ and $\alpha_2 = \varepsilon_2 - \varepsilon_3$. Then,

$$s_{\alpha_1}(\varepsilon_1) = \varepsilon_1 - \frac{2\langle \varepsilon_1, \varepsilon_1 - \varepsilon_2 \rangle}{\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle}(\varepsilon_1 - \varepsilon_2) = \varepsilon_1 - (\varepsilon_1 - \varepsilon_2) = \varepsilon_2,$$

$$s_{\alpha_1}(\varepsilon_2) = \varepsilon_2 - \frac{2\langle \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle}{\langle \varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_2 \rangle}(\varepsilon_1 - \varepsilon_2) = \varepsilon_2 + (\varepsilon_1 - \varepsilon_2) = \varepsilon_1.$$

SO,

$$s_{\alpha_1} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ \varepsilon_2 & \varepsilon_1 & \varepsilon_3 \end{bmatrix},$$

$$s_{\alpha_2} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ \varepsilon_1 & \varepsilon_3 & \varepsilon_2 \end{bmatrix}.$$
\[ \alpha_1 + \alpha_2 = \varepsilon_1 - \varepsilon_3 \]
Symmetric Group

A *subgroup* of $S_n$ is any subset of permutations that is closed under the composition operation.

One special way this can happen is by taking a *subset of the generators*, called a *parabolic subgroup*. For example, $S_4$ is a parabolic subgroup of $S_5$:

$$
\begin{array}{cccc}
\bullet s_1 & \bullet s_2 & \bullet s_3 & \bullet s_4 \\
\end{array}
$$

These are all the permutations in which the last entry 5 *is fixed*.

$$
S_4 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\
* & * & * & * & 5 \\
\end{bmatrix}
$$
If we wanted to use our understanding of $S_4$ to understand $S_5$, we could specify permutations in $S_5$ by

1. Permute the first four entries.
2. Move the entry 5 into its final position.

For example, we could build $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{bmatrix}$ as

$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \xrightarrow{s_1} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{bmatrix} \xrightarrow{s_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{bmatrix} \xrightarrow{s_4} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{bmatrix} \xrightarrow{s_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{bmatrix}$
Affine Symmetric Group

The affine symmetric group $\tilde{S}_n$ is presented as a Coxeter group by:

- Generators $s_0, s_1, \ldots, s_{n-1}$, with $s_i^2 = id$,
- Commuting relations $s_is_j = id$ if $|i - j| \geq 2$,
- Braid relations $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$ and $s_0s_{n-1}s_0 = s_{n-1}s_0s_{n-1}$.

This is an infinite Coxeter group, but notice that it has finite $S_n$ as a parabolic subgroup.
We can again view the generators $s_i$ as \textbf{reflections} of a vector space with orthonormal basis $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$. The formula for reflection by $s_1, s_2, \ldots, s_{n-1}$ is exactly the same as before:

- For $1 \leq i \leq n - 1$, let $s_i$ be the reflection that interchanges $\varepsilon_i$ and $\varepsilon_{i+1}$.

The reflection by $s_0$ is an \textbf{affine reflection} defined on $v = \sum_{j=1}^{n} a_j \varepsilon_j$ by

$$s_0(v) = (a_n + 1)\varepsilon_1 + a_2 \varepsilon_2 + \cdots + a_{n-1} \varepsilon_{n-1} + (a_1 - 1) \varepsilon_n.$$
\[ \alpha_1 = \varepsilon_1 - \varepsilon_2 \]
\[ \alpha_2 = \varepsilon_2 - \varepsilon_3 \]
Affine Symmetric Group

The simple roots $\Delta$ of type $A_{n-1}$ are

$$\alpha_1 = \varepsilon_1 - \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_3, \quad \ldots, \quad \alpha_{n-1} = \varepsilon_{n-1} - \varepsilon_n.$$  

The $\mathbb{Z}$-span $\Lambda_R$ of $\Delta$ is called the root lattice of type $A_{n-1}$. Note that these are just vectors whose coordinates sum to 0.

Recall that $S_n$ is a parabolic subgroup of $\widetilde{S}_n$. It turns out that there is a unique way to write any affine permutation as a pair

$$\left( \text{element of the root lattice}, \text{finite permutation} \right)$$

However, it’s better to look at all the affine permutations that correspond to a given root lattice point, and choose a special one to represent the root lattice point. This affine permutation called a minimal length coset representative.
Affine Symmetric Group

We want to study

minimal length coset representatives

↔ integer vectors whose coordinates sum to 0

↔ \( n \)-cores

↔ abacus diagrams

and especially, **how to project an \( n \)-core to an \((n - 1)\)-core?**

E.g.

\[
\begin{array}{c}
\bullet 0 \\
\bullet 1 \quad \bullet 2 \quad \bullet 3 \\
\end{array}
\begin{array}{c}
\longrightarrow
\\
\begin{array}{c}
\bullet 0 \\
\bullet 1 \quad \bullet 2 \\
\end{array}
\end{array}
\]
Core Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a partition and $n \geq 2$ be an integer.

**Example**

The $n$-residue of a box $(i, j)$ is the least nonnegative integer $\equiv j - i \mod n$. 
Core Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a partition and $n \geq 2$ be an integer.

Example

The $n$-residue of a box $(i,j)$ is the least nonnegative integer $\equiv j - i \mod n$. 
Core Notation

Let $\lambda = (\lambda_1 \geq \ldots \geq \lambda_r)$ be a partition and $n \geq 2$ be an integer.

Example

The hook length of a box $(i, j)$ is the number of boxes to the right and below the box, including itself. It is denoted $h_{(i,j)}^{\lambda}$.
Core Notation

Let \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_r) \) be a partition and \( n \geq 2 \) be an integer.

**Example**

<table>
<thead>
<tr>
<th>14</th>
<th>10</th>
<th>7</th>
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The *hook length* of a box \((i, j)\) is the number of boxes to the right and below the box, including itself. It is denoted \( h_{(i,j)}^\lambda \).
Cores

Definition

A partition $\lambda$ is an $n$-core if $n \nmid h_{(i,j)}^\lambda$ for every box $(i,j)$ of $\lambda$.

Example

\[
\begin{array}{ccccccc}
14 & 10 & 7 & 6 & 5 & 3 & 2 & 1 \\
10 & 6 & 3 & 2 & 1 \\
6 & 2 \\
5 & 1 \\
3 \\
2 \\
1
\end{array}
\]

$\lambda$ is a 4-core.
**Question**

Given an $n$-core, how can we project to obtain an $(n - 1)$-core?

**Example**

<table>
<thead>
<tr>
<th>4-core</th>
<th>3-core</th>
<th>2-core</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>5</td>
<td>1</td>
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<tr>
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$\rightarrow$

| 5      | 2      | 1      |
| 2      |        |        |
| 1      |        |        |

$\rightarrow$

| 1      |        |        |
Cores

$n$-core partitions index:

- Schubert cells in the affine Grassmannian $\text{Gr}$ of $SL(n, \mathbb{C})$.
  $(\text{Gr} \cong SL_n(\mathbb{C}((t))))/SL_n(\mathbb{C}[[t]]).$]
- $k$-Schur functions and dual $k$-Schur functions in $H_*(\text{Gr}) \cong \Lambda_n$ and $H^*(\text{Gr}) \cong \Lambda^n$, respectively.
- Blocks in the representation theory of the symmetric group $S_k$ over a field of characteristic $n > 0$. 
Cores

- $C_n = \text{The set of all } n\text{-cores}.$
- $C^k_n = \text{The subset of } C_n \text{ having first part } k.$
- $C^\leq_k n = \text{The subset of } C_{n-1} \text{ having first part } \leq k.$

We will define a bijection

$$\Phi^k_n : C^k_n \to C^\leq_k n_{n-1}$$

Then,

$$\sum_{k \geq 0} |C^k_n| x^k = \sum_{k \geq 0} \binom{k+n-2}{k} x^k = \frac{1}{(1-x)^{n-1}}.$$

(Proof:

$$\binom{k+n-2}{k} = \binom{k+n-3}{k} + \binom{k+n-4}{k-1} + \cdots + \binom{n-3}{0}.$$
The partition shape is determined by first column hooklengths. These can be generalized to $\beta$-numbers.
## Beta numbers and Abaci

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<tr>
<th>Runner</th>
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### 3-core

```
... -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 ...
... o o o o ● o o o ● o o ● o ● o ● o ● o ● o ● ...
```
The abacus for $\beta = (8, 5, 4, 2, 1, -1, -2, -3, \ldots)$ has balance number $2 = (-1) + 1 + 2$. 
The abacus for $\beta = (8, 5, 4, 2, 1, -1, -2, -3, \ldots)$ has balance number 2. The abacus for $\beta = (9, 6, 5, 3, 2, 0, -1, -2, \ldots)$ has balance number $3 = 3 + (-1) + 1$. 
Beta numbers and Abaci

Theorem

Theorem 2.7.16, Lemma 2.7.38 in James–Kerber

- $\lambda$ is an $n$-core if and only if any (equivalently, every) abacus of $\lambda$ on $n$ runners is flush.

- Moreover, in the **balanced flush abacus** of an $n$-core $\lambda$, each active bead on runner $i$ corresponds to a row of $\lambda$ whose rightmost box has residue $i$. 
Beta numbers and Abaci

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<tr>
<th>Level</th>
<th>Runner 0</th>
<th>Runner 1</th>
<th>Runner 2</th>
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3-core

\[\begin{array}{cccc}
0 & 1 & 2 & 0 \\
2 & 0 &   &   \\
1 & 2 &   &   \\
0 &   &   &   \\
2 &   &   &   \\
\end{array}\]
The bijection $\Phi_n^k$

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4-core (8, 5, $2^2$, $1^3$)

$\Phi_4^8 \rightarrow$

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3-core (2, $1^2$).
The bijection $\Phi^k_n$

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4-core $(8, 5, 2^2, 1^3)$  \[ \Phi^8_4 \]  3-core $(2, 1^2)$. 

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The bijection $\Phi^n_k$

Let $a = (a_1, \ldots, a_n) \in \Lambda_R$ written in the $\varepsilon_i$ basis, so each $a_i \in \mathbb{Z}$ and $\sum_{i=1}^n a_i = 0$.

We form a balanced flush abacus from $a$ by filling the $(i - 1)^{st}$ runner with beads from $-\infty$ down to level $a_i$.

This defines a bijection

$$\pi : \{(a_1, \ldots, a_n) : a_i \in \mathbb{Z}, \sum_{i=1}^n a_i = 0\} \rightarrow \{\text{balanced flush abaci}\} \rightarrow C_n.$$
The bijection $\Phi^k_n$

Example

$n = 4$, $(2, 0, 0, -2)$ corresponds to

\[
\begin{array}{cccc}
-8 & -7 & -6 & -5 \\
-4 & -3 & -2 & -1 \\
0 & 1 & 2 & 3 \\
4 & 5 & 6 & 7 \\
8 & 9 & 10 & 11 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 & 0 \\
3 & 0 \\
2 \\
1 \\
0 \\
\end{array}
\]
The bijection $\Phi_n^k$

Proposition

Suppose that $\pi(a) = \pi(a_1, \ldots, a_n) = \lambda$. Then we have

$$\lambda_1 = (a_i - 1)n + i$$

where $a_i$ is the rightmost occurrence of the largest coordinate in $a$.

Corollary

For $k \geq 0$, let $H_n^k$ denote the affine hyperplane

$$H_n^k = \{a = (a_1, \ldots, a_n) \in \mathbb{R}^n : (a, \varepsilon(k \mod n)) = \left\lceil \frac{k}{n} \right\rceil \} \cap V$$

inside $V$, where $1 \leq (k \mod n) \leq n$. Then under the correspondence $\pi$, the $n$-cores $\lambda$ with $\lambda_1 = k$ all lie inside $H_n^k \cap \Lambda_R$. 
The bijection $\Phi^n_k$

7 = $\lambda_1 = (a_i - 1)n + i = (2 - 1)4 + 3$.

$H_4^{7} = \{(a_1, a_2, a_3, a_4) : a_3 = 2\} \cap V$
The bijection $\Phi^k_n$

**Theorem**

Let $\psi_n$ be the affine map defined by

$$\psi_n(a_1, \ldots, a_n) = (a_n + 1, a_1, a_2, \ldots, a_{n-1}).$$

Then,

$$\pi^{-1} \circ \Phi^k_n \circ \pi(a_1, \ldots, a_n) = \psi_{n-1}^{a_i}(a_1, \ldots, \hat{a}_i, \ldots, a_n)$$

where $a_i$ is the rightmost occurrence of the largest entry among

$\{a_1, \ldots, a_n\}$ and the circumflex indicates omission.
The bijection $\Phi^k_n$

We can factor this map into **translation** composed with **root system embedding**.

**Example**

Let $n = 3$. The affine hyperplane $H_3^7$ contains the partition $\pi(3, 1, -4) = (7, 5, 4^2, 3^2, 2^2, 1^2)$. Translation by $t = (-3, 1, 2)$ sends $H_3^7$ to

$$\left\{ (a_1, a_2, a_3) \in V : a_1 = 0 \right\}$$

and in particular sends $(3, 1, -4)$ to $(0, 2, -2)$.

We view this as a subspace of $\mathbb{R}^2$ with orthonormal basis $\{e'_1, e'_2\}$ and $A_{n-2}$ root system. The embedding identifies $e'_1$ with $e_3$ and $e'_2$ with $e_2$ and we have $\psi^3(1, -4) = (-2, 2)$ corresponding to $\Phi^7_3(7, 5, 4^2, 3^2, 2^2, 1^2) = (4, 3, 2, 1)$. 
Open questions:

- How do these combinatorics generalize to other reflection groups?
- What does the projection $\Phi^k_n$ imply about cells in the affine Grassmannian, $k$-Schur functions, or blocks in $S_n$-modules?