

Math 108 Summary of cardinality

Numbers in parentheses refer to the Yellow Book.

Definition. Two sets A and B are said to be *equivalent* if there exists a bijection $f : A \rightarrow B$. We write $A \approx B$.

Theorem (Thm. 20.1). *Equivalence is an equivalence relation.*

Definition. A set A is said to be *finite* if either $A = \emptyset$ or $A \approx \{1, \dots, n\}$ for some $n \in \mathbf{Z}^+$; otherwise, A is *infinite*.

Theorem (Cor. 20.11). *If S is a finite set, and $A \subseteq S$, then A is finite.*

Theorem (Thm. 20.12). *If A and B are finite sets, then $A \cup B$ is finite.*

Theorem (Cor. 20.15). *If A and B are finite sets, then $A \times B$ is finite.*

Theorem (Thm. 21.2, a.k.a. the Pigeonhole Principle). *If $m, n \in \mathbf{Z}^+$, $m > n$, and $f : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$ is a function, then f is not one-to-one.*

Corollary (Thm. 21.3). *The set \mathbf{N} is infinite.*

Definition. If $A \approx \{1, \dots, n\}$ for $n \in \mathbf{Z}^+$, we say that $|A| = n$. (This is well-defined only because of the Pigeonhole Principle.)

Definition. A set A is said to be *countably infinite* if $A \approx \mathbf{N}$. A set A is said to be *countable* if A is either finite or countably infinite; otherwise, A is *uncountable*.

Lemma (Ex. 22.5/the countability lemma). *Let A be a nonempty set. Then A is countable if and only if there exists an injective function $f : A \rightarrow \mathbf{N}$.*

Theorem (Cor. 22.4). *If S is a countable set, and $A \subseteq S$, then A is countable.*

Theorem (Thm. 22.6). *If A and B are countable sets, then $A \cup B$ is countable.*

Theorem (Cor. 22.10). *If A and B are countable sets, then $A \times B$ is countable.*

Theorem (Thm. 22.13). *If A_n is a countable set for each $n \in \mathbf{Z}^+$, then $\bigcup_{n \in \mathbf{Z}^+} A_n$ is a countable set.*

Theorem (Thm. 22.11). *The set of rational numbers, \mathbf{Q} , is countably infinite.*

Theorem (Thm. 22.12). *The set of real numbers, \mathbf{R} , is uncountable.*

Proof: Cantor diagonalization.