Sample Exam 2
Math 108, Spring 2008

1. (14 points) Let S be a nonempty subset of $\mathbf{R}$. Define what it means for a real number L to be an infimum of S.

In questions 2–4, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

2. (12 points) It is possible that there exists an equivalence relation $\sim$ on $\mathbf{Z}$ such that $E_2 = \{1, 2, 3, 7\}$ and $E_{-4} = \{-4, 1, 7\}$. (Recall that E_n is the equivalence class of $n \in \mathbf{Z}$.)

3. (12 points) If $f : \mathbf{R} \to \mathbf{R}^+$ and $g : \mathbf{R}^+ \to \mathbf{R}$ are functions such that $g(f(x)) = x$ for all $x \in \mathbf{R}$ and $f(g(y)) = y$ for all $y \in \mathbf{R}^+$ (i.e., f and g are inverses of each other), then f must be one-to-one and g must be onto.

4. (12 points) If $f : X \to Y$ is a well-defined function, $x_1, x_2 \in X$, $y \in Y$, $f(x_1) = y$, and $f(x_2) = y$, then it must be the case that $x_1 = x_2$.

5. (16 points) **OUTLINE** the proof of the following theorem. More precisely, state the assumption and the conclusion for the proof of the theorem, and work forwards and backwards towards the middle as much as you can by only applying definitions. **DO NOT TRY TO PROVE THIS THEOREM.**

> **Theorem.** Let
>
> $$A = \left\{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\right\},$$
> $$B = \left\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\right\},$$
>
> and let the function $f : A \to B$ be defined by the formula
>
> $$f(x, y, z) = (x, y)$$
>
> Then f is a bijection.

6. (16 points) **PROOF QUESTION.** We define a relation $\sim$ on $\mathbf{R}$ by saying that $x \sim y$ if and only if $x = ry$ for some rational number r such that $r \neq 0$ (i.e., $r \in \mathbf{Q}$, $r \neq 0$). Prove that $\sim$ is an equivalence relation.

7. (18 points) **PROOF QUESTION.** Let S be a bounded subset of $\mathbf{R}$, and define

$$T = \{7x \mid x \in S\}.$$

Let $u = \sup S$.

(a) Prove that T is bounded above.

(b) Prove that $\sup T$ exists and that $\sup T \leq 7u$.