Sample Final Exam
Math 108, Spring 2008

1. (16 points) Let (x_n) be a sequence, and let L be a real number. Define what it means to say that $\lim_{n\to\infty} x_n = L$.

2. (16 points) Let X and Y be sets, let $f : X \to Y$ be a function, let A be a subset of X, and let B be a subset of Y. Define each of the following objects that is always defined under these circumstances.

- $f(A)$
- $f(B)$
- $f^{-1}(A)$
- $f^{-1}(B)$

In question 3–6, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

3. (14 points) Let X and Y be sets, and let $f : X \to Y$ be a function such that if $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$, then $x_1 = x_2$. Then it must be the case that f is invertible.

4. (14 points) Let X be a nonempty set, and let $\sim$ be an equivalence relation on X. Recall that for $x \in X$, E_x denotes the equivalence class of x under $\sim$. For $y, z \in X$, if $y \not\sim z$, then it must be the case that $E_y \cap E_z = \emptyset$.

5. (14 points) For each $i \in \mathbf{Z}^+$, let A_i be a nonempty subset of $\mathbf{R}$, and suppose that $A_i \supseteq A_{i+1}$ for each $i \in \mathbf{Z}^+$ (i.e., $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$). Then it must be the case that

$$\bigcap_{i=1}^{\infty} A_i \neq \emptyset.$$

6. (14 points) Let S be an infinite set. Then it must be the case that S is equivalent to Z.

7. (18 points) **PROOF QUESTION.** Define a function $g : \mathbf{R}^2 \to \mathbf{R}^2$ by the formula

$$g(x, y) = (x + y, y)$$

for all $x, y \in \mathbf{R}^2$. Prove that g is a bijection.

8. (18 points) **PROOF QUESTION.** Let

$$A = \left\{(x, y, z) \in \mathbf{R}^3 \mid x = y\right\},$$
$$B = \left\{(x, y, z) \in \mathbf{R}^3 \mid x = 2z\right\},$$
$$C = \left\{(x, y, z) \in \mathbf{R}^3 \mid (x - y)^2 + (y - 2z)^2 = 0\right\}.$$

Prove that $A \cap B \subseteq C$.

9. (18 points) **PROOF QUESTION.** Let

$$S = (-2, 3] = \{x \in \mathbf{R} \mid -2 < x \leq 3\}.$$

There are two options for this problem, a full credit version, and an easier partial credit version.

- (Full credit version) Prove that $\inf S = -2$.
- (Partial credit version) Prove that $\sup S = 3$.

If you attempt both versions, you will receive the larger of the two scores as your score for this problem.

10. (18 points) **PROOF QUESTION.** Recall that if A_k is a set for all $k \in \mathbf{Z}^+$, then

$$\bigcup_{k=1}^{1} A_k = A_1,$$

$$\bigcup_{k=1}^{n+1} A_k = \left(\bigcup_{k=1}^{n} A_k\right) \cup A_{n+1} \qquad \text{for } n \geq 1.$$

(In fact, we can take this as a definition of $\bigcup_{k=1}^{n} A_k$.)

Suppose that for all $k \in \mathbf{Z}^+$, A_k is a nonempty subset of $\mathbf{R}$ that is bounded above. Prove, by induction on n, that for each $n \in \mathbf{Z}^+$, the set $\bigcup_{k=1}^{n} A_k$ is bounded above.

11. (20 points) **PROOF QUESTION.** Let (x_n) be a sequence such that $\lim_{n\to\infty} x_n = 0$, and let (y_n) be a bounded sequence of nonzero real numbers. Prove that $\lim_{n\to\infty} x_n y_n = 0$.

12. (20 points) **PROOF QUESTION.** Recall that a *rational-valued sequence* is a function $f : \mathbf{Z}^+ \to \mathbf{Q}$, usually written (a_n), where n ranges over all positive integers. Let

$$S_k = \{(a_n) \mid (a_n) \text{ is a rational sequence, } a_n = 0 \text{ for all } n > k\}.$$

In other words, let S_k be the set of all rational-valued sequences that take the value 0 for all terms after term number k.

Finally, we say that a rational sequence (a_n) has *finite support* if there exists some $k \in \mathbf{N}$ such that $a_n = 0$ for all $n > k$.

(a) Prove that S_2 is equivalent to $\mathbf{Q} \times \mathbf{Q}$.

(b) Prove that for any $k \in \mathbf{Z}^+$, S_k is a countable set.

(c) Let S be the set of all rational sequences with finite support. Prove that S is countable. (Suggestion: How does S relate to the sets S_k?)