Math 126, problem set 06
Outline due: Wed Oct 21
Due: Mon Oct 26
Last revision due: Mon Nov 16

Review of calculus:

Infinite series and their convergence. An *infinite series* is an "infinite sum" like

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots.$$

To make this idea precise, we define the *partial sums* of the above series to be

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \dots + a_n, \\ &\vdots \end{aligned}$$

In other words, S_n is what you get when you add up the first n terms of the sum.

If the limit $\lim_{n\to\infty} S_n$ exists, then we say that the series $\sum_{n=1}^{\infty} a_n$ *converges*, and we define the sum of the infinite series to be

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n.$$

Otherwise, we say that $\sum_{n=1}^{\infty} a_n$ *diverges* (i.e., its sum is not defined).

The integral test. If the terms a_n of the series $\sum_{n=1}^{\infty} a_n$ come from some decreasing continuous function $f(x) \geq 0$, i.e., $a_n = f(n)$, then we have the following theorem:

Theorem (Integral test)**.** *Fix c such that $f(x)$ is decreasing, continuous, and non-negative for $c \leq x < \infty$. The series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} f(n)$ converges if and only if the improper integral*

$$\int_c^{\infty} f(x)\,dx = \lim_{d\to\infty} \int_c^d f(x)\,dx$$

converges (i.e., is finite).

The prime number theorem, paraphrased. Recall that $\pi(x)$ is the number of primes $\leq x$, and p_n is the nth prime. One way to think of the prime number theorem is:

Theorem (Paraphrased Prime Number Theorem)**.** *For large x, $\pi(x)$ is approximately equal to $\dfrac{x}{\ln x}$ (percentagewise). Equivalently, for large n, p_n is approximately equal to $n \ln n$ (percentagewise).*

Problems to be turned in:

1. (a) Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots.$$

   With the aid of a calculator, spreadsheet, or computer, find the sum of the first 50 terms of this series, then the first 100, and the first 200. (Or at least go out to some reasonable extent given the method you are using.)

   (b) Use the integral test to determine if the series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

   (c) Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{p_n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots,$$

   i.e., the sum of the reciprocals of the primes. A list of the first 500 primes can be found on the course homepage, both in text and in data form. With the aid of a calculator, spreadsheet, or computer, find the sum of the first 50 terms of this series, then the first 100, and the first 200. (Or at least go out to some reasonable extent given the method you are using.) Does it seem like the series converges or diverges?

   (d) Use the approximation for p_n in the prime number theorem to try to determine evidence as to whether the series $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges or diverges. (What does the integral test indicate?)

2. (a) Let n be an odd positive integer. Find a formula for $(x^n + 1)/(x + 1)$, and prove your formula.

   (b) 14.1.

3. 15.2.

4. Let n be a positive integer. The *divisor lattice* for n is a graph (i.e., vertices connected by lines) drawn in the plane according to the following rules:

   - The graph has exactly one vertex for every divisor of n, labelled by that divisor.
   - If a, b are divisors of n, and a divides b, then vertex b is drawn above vertex a.
   - The vertex a is connected to the vertex b if and only if either $b = ap$ for some prime p or $a = bp$ for some prime p.

(a) Draw the divisor lattices for $n = 5$, 25, 125, and 625. What does the divisor lattice for 5^a look like?

(b) Draw the divisor lattices for $n = 6$, 12, 18, 36. What does the divisor lattice for $2^a 3^b$ look like?

(c) Draw the divisor lattices for $n = 2$, 6, 30, 210. Do you see a pattern? Generalize.

5. Assume $m, n \geq 2$ and $a > 0$ throughout this problem.

   (a) Prove that if c divides m and d divides n, then cd divides mn.

   (b) Find a, m, n such that a divides mn, and that there exists more than one pair c, d such that c divides m, d divides n, and $a = cd$.

   (c) Prove that if $\gcd(m, n) = 1$ and a divides mn, then there exists *exactly* one pair c, d such that c divides m, d divides n, and $a = cd$. (Suggestion: Use FTA.)

   (d) Prove that if $\gcd(m, n) = 1$, then $\sigma(mn) = \sigma(m)\sigma(n)$.