Sample Final Exam
Math 128A, Spring 2009

1. (14 points) Define what an isomorphism of groups is, and define what it means for two groups G and H to be isomorphic.

2. (18 points) Let R be a ring (not necessarily a commutative ring), and let A be an ideal of R. Define the factor ring R/A. In particular, describe the **elements** of R/A, the **operations** in R/A, and the **zero element** of R/A.

3. (12 points) Let

$$\alpha = (1\ 2\ 5\ 7\ 8)(3\ 6\ 10)(4\ 11),$$
$$\beta = (1\ 10\ 4\ 9\ 3\ 2\ 5\ 8\ 7).$$

(a) Calculate α^{-1} and $\alpha\beta$. Put your final answers in cycle form.

(b) Calculate the order of $\alpha\beta$.

No explanation necessary, but show all your work.

4. (12 points) Let $G = \langle a \rangle$, where $\langle a \rangle = 18$ (i.e., the order of a is 18).

(a) List all generators of G. (I.e., list all $b \in G$ such that $G = \langle b \rangle$.)

(b) Find the order of $\langle a^8 \rangle$.

No explanation necessary, but show all your work.

For questions 5–10, you are given a statement. If the statement is true, you need only write "True", though a justification may earn you partial credit if the correct answer is "False". If the statement is false, write "False", and justify your answer **as specifically as possible**. (Do not just write "T" or "F", as you may not receive any credit; write out the entire word "True" or "False".)

5. (12 points) **(TRUE/FALSE)** If G is a nonabelian group, $a, b \in G$, $a \neq e$, and $b \neq e$, then it must be the case that $ab \neq ba$.

6. (12 points) **(TRUE/FALSE)** It is possible to find a positive integer n such that there are precisely 6 abelian groups of order n, up to isomorphism.

7. (12 points) **(TRUE/FALSE)** Let R be a commutative ring with unity element 1. If $a \in R$ and $a^2 + a = 0$, then it must be the case that either $a = 0$ or $a = -1$.

8. (12 points) **(TRUE/FALSE)** The group $U(12)$ is cyclic.

9. (12 points) **(TRUE/FALSE)** Let

$$H = \{\sigma \in S_5 \mid \sigma(1) = 2\}.$$

In other words, let H be the set of all elements of S_5 that send 1 to 2. Then H is a subgroup of S_5.

10. (12 points) **(TRUE/FALSE)** Let R be the ring

$$R = \left\{ a + b\sqrt{2} \,\middle|\, a, b \in \mathbf{Z}_5 \right\}.$$

Then R is a field.

11. (18 points) **PROOF QUESTION.** Let R be a noncommutative ring, let a be an element of R, and let

$$C(a) = \{ x \in R \mid xa = ax \}.$$

Prove that $C(a)$ is a subring of R.

12. (18 points) **PROOF QUESTION.** Let $G = \mathbf{Z}_6 \oplus \mathbf{Z}_{10}$, and let

$$N = \{(0,0), (2,0), (4,0), (0,5), (2,5), (4,5)\} \leq G.$$

You may take it as given that N is a subgroup of G.

(a) Explain how you know that N is normal in G.

(b) Prove that $G/N \approx \mathbf{Z}_{10}$.

13. (18 points) **PROOF QUESTION.** Recall that if H is a group, then $Z(H)$ is the set of all $z \in H$ such that $az = za$ for all $a \in H$.

Let $\varphi : G \to \overline{G}$ be a homomorphism from G onto $\overline{G}$ (i.e., assume φ is a surjective homomorphism). Prove that if $z \in Z(G)$, then $\varphi(z) \in Z(\overline{G})$.

14. (18 points) **PROOF QUESTION.** Let G be a group of order 30. Prove that G must contain either an element of order 2 or an element of order 3. (Suggestion: What happens if G contains no elements of order 2 and no elements of order 3?)