

Math 128a final, Fall 2005

1. Define what an isomorphism of groups is, and define what it means for two groups G and H to be isomorphic.

2. Let R be a ring (not necessarily a commutative ring), and let A be an ideal of R . Define the factor ring R/A . In particular, describe the **elements** of R/A , the **operations** in R/A , and the **zero element** of R/A .

3. Let

$$\alpha = (1\ 2\ 5\ 7\ 8)(3\ 6\ 10)(4\ 11),$$
$$\beta = (1\ 10\ 4\ 9\ 3\ 2\ 5\ 8\ 7).$$

(a) Calculate α^{-1} and $\alpha\beta$. Put your final answers in cycle form.

(b) Calculate the order of $\alpha\beta$.

4. Let $G = \langle a \rangle$, where $\langle a \rangle = 18$ (i.e., the order of a is 18).

(a) List all generators of G . (I.e., list all $b \in G$ such that $G = \langle b \rangle$.)

(b) Find the order of $\langle a^8 \rangle$.

5. (**TRUE/FALSE**) If G is a nonabelian group, $a, b \in G$, $a \neq e$, and $b \neq e$, then it must be the case that $ab \neq ba$.

6. (**TRUE/FALSE**) It is possible to find a positive integer n such that there are precisely 6 abelian groups of order n , up to isomorphism.

7. (**TRUE/FALSE**) Let R be a commutative ring with unity element 1. If $a \in R$ and $a^2 + a = 0$, then it must be the case that either $a = 0$ or $a = -1$.

8. (**TRUE/FALSE**) The group $U(12)$ is cyclic.

9. (**TRUE/FALSE**) Let

$$H = \{\sigma \in S_5 \mid \sigma(1) = 2\}.$$

In other words, let H be the set of all elements of S_5 that send 1 to 2. Then H is a subgroup of S_5 .

10. (**TRUE/FALSE**) Let R be the ring

$$R = \left\{ a + b\sqrt{2} \mid a, b \in \mathbf{Z}_5 \right\}.$$

Then R is a field.

11. **PROOF QUESTION.** Let R be a noncommutative ring, let a be an element of R , and let

$$C(a) = \{x \in R \mid xa = ax\}.$$

Prove that $C(a)$ is a subring of R .

12. PROOF QUESTION. Let $G = \mathbf{Z}_6 \oplus \mathbf{Z}_{10}$, and let

$$N = \{(0, 0), (2, 0), (4, 0), (0, 5), (2, 5), (4, 5)\} \leq G.$$

You may take it as given that N is a subgroup of G .

- (a) Explain how you know that N is normal in G .
- (b) Prove that $G/N \approx \mathbf{Z}_{10}$.

13. PROOF QUESTION. Recall that if H is a group, then $Z(H)$ is the set of all $z \in H$ such that $az = za$ for all $a \in H$.

Let $\varphi : G \rightarrow \overline{G}$ be a homomorphism from G onto \overline{G} (i.e., assume φ is a surjective homomorphism). Prove that if $z \in Z(G)$, then $\varphi(z) \in Z(\overline{G})$.

14. PROOF QUESTION. Let G be a group of order 30. Prove that G must contain either an element of order 2 or an element of order 3. (Suggestion: What happens if G contains no elements of order 2 and no elements of order 3?)